Research Article

Digital Transformation of Optimal Decision-making in Economic and Engineering Systems Based on the Theory and Methods of Vector Optimization

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Abstract

Objective: The purpose of the work is to analyze, study the processes of digital transformation and make optimal decisions in economic and engineering systems based on the theories and methods of vector optimization.

Methods: Methods for studying the processes of digital transformation of economic and engineering systems are based on the author’s theory of vector optimization, which is represented by: axiomatics and optimality principles that show how one solution is better than another. On the basis of the principles of optimality, constructive methods for solving vector problems have been built, firstly, with equivalent criteria, and, secondly, with a given priority of the criterion: https://rdcu.be/bhZ8i (Springer Nature’s “Vector optimization with equivalent and priority criteria” is distributed free of charge.)

The paper presents that the problem of accepting the optimal process is carried out as a result of the digital transformation of the initial (statistical, experimental) data into vector problems in optimization, which represents a mathematical model of economic and engineering systems. Depending on the number and type of criteria, we solve problems with a vector of criteria, each component of which is directed to the maximum, and with a vector of criteria, each component of which is aimed at a minimum, in aggregate.

Results: The result of the study is the practical orientation of digital transformation, which is shown in the computer-aided design of economic and engineering systems based on vector optimization. For this purpose, software for solving vector problems of linear and nonlinear programming has been developed. Linear vector problems are used in economic systems (forecasting, planning). Software for solving vector problems of nonlinear programming are used digital transformation of optimal decision-making in engineering problems.

Conclusion: In conclusion, numerical examples in the field of economic systems are presented by the digital transformation of optimal decision-making in the development of the company. When making optimal decisions in engineering systems, the following has been developed: the construction of initial data (technical task) for modeling; transformation of a mathematical model under uncertainty into a model under conditions of certainty; making an optimal decision with equivalent criteria; making the optimal decision with a given priority criterion.
Keywords: digital transformation, theory of vector optimization, methods of vector optimization, modeling of economic systems, modeling of engineering systems


1 INTRODUCTION

Research in the framework of the digital transformation of the development of economic and engineering systems showed that their development depends on a certain set of functional characteristics that need to be taken into account at the design stage. When analysing the functioning of both economic and engineering systems, it turned out that an improvement in one of the characteristics leads to a deterioration in other characteristics. To improve the functioning of the system, it is necessary to improve all the characteristics in the aggregate. The mathematical model of such (economic, technical) is represented by multi-criteria optimization problems and, as a result, the solution of multi-criteria (vector) problems of mathematical programming is required. Research on this class of problems began more than a hundred years ago in Pareto\textsuperscript{[1]}\textsuperscript{[1]}. Further research on multi-criteria optimization was carried out both at the theoretical level by foreign\textsuperscript{[2]}\textsuperscript{[2]} and Russian authors\textsuperscript{[3-6]}\textsuperscript{[3-6]}, and on solving practical problems first in the field of engineering systems\textsuperscript{[7-23]}\textsuperscript{[7-23]}, mathematical problems\textsuperscript{[24-30]}\textsuperscript{[24-30]} and then in economics\textsuperscript{[31-47]}\textsuperscript{[31-47]}.

The purpose of the work is to analyze and study the processes of digital transformation of the development of economic and engineering systems on the basis of the theory and methods of vector optimization. Within the framework of the theory of vector optimization, the principles of optimality of solving vector problems with equivalent criteria and with a given priority of the criterion are presented, and constructive methods for solving vector optimization problems are shown. In the applied part, we presented constructive methods of digital transformation of economic systems in the form of vector problems of linear programming. For modeling and digital transformation of engineering systems, vector problems of nonlinear programming were used, which were solved under conditions of certainty and uncertainty.

To realize this goal, the work presents two parts: theoretical and applied parts. The first part presents the construction of a vector problem of mathematical programming\textsuperscript{[5-21]}\textsuperscript{[5-21]}. Within the framework of the theory of vector optimization, the principles of optimality of solving vector problems with equivalent criteria and with a given priority of the criterion are presented. On the basis of the principles of optimality, constructive methods for solving vector optimization problems have been developed, which make it possible to make an optimal decision, firstly, with equivalent criteria, and secondly, with a given priority of the criterion. In the study of the problem of vector optimization, a numerical solution of the vector problem of linear programming (VPLP) is presented, as well as a numerical solution of the vector problem of nonlinear (convex) programming with four homogeneous criteria.

In the second applied part of the work, the solution of two classes of problems in the fields of economic and engineering systems is presented.

In the field of economic systems\textsuperscript{[31-45]}\textsuperscript{[31-45]} formed constructive methods for solving vector problems of linear programming are used to model (forecast, plan) the development of the company\textsuperscript{[40-42]}\textsuperscript{[40-42]}, the market\textsuperscript{[39,43]}\textsuperscript{[39,43]}, the region\textsuperscript{[43,44]}\textsuperscript{[43,44]} and the state as a whole\textsuperscript{[44,45]}\textsuperscript{[44,45]}. The paper shows the modeling of the development of the company, based on software. The text of the solution of the VPLP is presented.

In the study of engineering systems, which include technical systems\textsuperscript{[10-13,17,18]}\textsuperscript{[10-13,17,18]}, technological processes\textsuperscript{[14,20,47]}\textsuperscript{[14,20,47]}, materials\textsuperscript{[16]}\textsuperscript{[16]}, the software developed by the author is used. The software allows solving vector problems of nonlinear programming in conditions of certainty and uncertainty. In the paper, the uncertainty represented by experimental data is transformed into conditions of certainty. The methodology of decision-making is developed, which includes: construction of the numerical model of the object in the form of a vector problem; solving the problem of decision-making with equivalent criteria; solution of the vector problem of decision-making with the priority of the criterion.

Main Designations:
\( N \) - set of natural numbers.
\( R \) - set of real numbers; numerical straight line.
\( R^n \) - arithmetic real \( n \) - dimensional space; Euclidean \( n \) - dimensional space;
\{a, b, c, x, y, \ldots\} - the set consisting of elements \(a, b, c, x, y, \ldots\)

\(\forall\) - generality quantifier: "for all".

\(\exists\) - existential quantifier: "exists".

\(\emptyset\) - empty set.

\(\varepsilon\) - sign of an accessor to a set.

\(\subset\) - sign of inclusion of a set.

\(A \cap B\) - a product of the sets of \(A\) and \(B\).

\(A \cup B\) - a union of the sets of \(A\) and \(B\).

\(X = \{x_1, \ldots, x_j, \ldots, x_N\}\) - the set consisting of \(N\) elements or \(X = \{x_j, j=1, \ldots, N\}\), or \(X = \{x_j, j = 1, N\}\), where \(j\) - number of the index (object), \(N\) - number (number of the last index), \(N\) - a set of indices.

\(\equiv\) - identically equal.

\(\text{Lim}\) - a limit.

\(\max_{\in X} x\) - the greatest (maximal) element of a great number of \(X\).

\(\min_{\in X} x\) - the least (minimum) element of a great number of \(X\).

\(\max_{\in X} f(X)\) - the greatest (maximal) value of function \(f\) on a great number of \(X\).

\(\min_{\in X} f(X)\) - the least (minimum) value of function \(f\) on a great number of \(X\).

\(\max_{\in X} f(X) \equiv F\) - the greatest (maximal) value of function \(f\) to which the functional value \(F\), on a great number of \(X\) is identically appropriated.

\(\min_{\in X} f(X) \equiv F\) - the least (minimum) value of function \(f\) to which the functional value \(F\), on a great number of \(X\) is identically appropriated.

\(\max F(X) = \{\max f_k(X), k = 1, K\}\) - vector criterion of maximizing with which each component is maximized, \(K\)-number, a \(K \equiv 1, K\) - a set of criterion indices.

2 MATERIALS AND METHODS

2.1 The Vector Problem of Mathematical Programming: Research, Analysis

The Vector problem of mathematical programming (VPMP) is a standard mathematical programming problem that has many criteria. In the problem, many criteria are represented as a vector of criteria. Vector problems are divided into uniform and non-uniform VPMP. A uniform maximizing Vector Mathematical Programming Problem is a vector problem in which each criterion is directed towards maximizing-max. A uniform minimizing Vector of Mathematical Programming Problem is a vector problem in which each criterion is directed towards minimizing-min. A non-uniform Vector Mathematical Programming Problem is a vector problem in which the set of criteria is shared between two subsets (vectors) of criteria (maximization - max and minimization - min, respectively), e.g., non-uniform vector mathematical of programming problem are associated with two types of uniform problems.

According to these definitions, we will present a vector problem of mathematical programming with non-uniform criteria\(^{[5,19,21]}\) in the following form:

\[
\begin{align*}
\text{Opt } F(X) = \{ \text{max } F_1(X), k = 1, K_1 \} \\
\text{min } F_2(X) = \{ \text{min } f_k(X), k = 1, K_2 \} \quad (1) \\
g(X) \leq B(3) \\
X \geq 0 \quad (4)
\end{align*}
\]

where \(X = \{x_j, j = 1, N\}\) is a vector of material variables, \(N\) - dimensional Euclidean space of \(R^N\), (designation \(j = 1, 2, \ldots, K\)); \(F(X)\) is a vector criterion (function) having \(K\) - a component function, \(K\) - set power \(K\), \(f(X), k = 1, K\). The set \(K\) consists of sets of \(K\), a criterion of maximization and \(K\) a criterion of minimization; \(K = K_1 \cup K_2\) therefore we enter the designation of the operation "opt," which includes maximum and minimum.

\(F_1(X) = \{f_k(X), k = 1, K_1\}\) is maximizing vector-criterion, \(K_1\) - number of criteria, and \(K_1 = 1, K_1\) is a set of maximizing criteria (a VPMP (1), (3), (4) represents a vector problem of mathematical programming with the homogenous maximizing criteria). Let's further assume that \(f_k(X), k = 1, K_1\) are the continuous concave functions.

\(F_2(X) = \{f_k(X), k = 1, K_2\}\) is vector criterion (function) in which each component is minimized, \(K_2 = 1, K_2\) - a set of minimization criteria, \(K_2\) - number, (the VPMP (2)- (4) are vector problems of mathematical programming with the homogenous minimization criteria). We assume that \(f_k(X), k = 1, K_2\) are the continuous convex functions (we will sometimes call these the minimization criteria), i.e., \(K_1 \cup K_2 = K\), \(K_1 \subset K\), \(K_2 \subset K\).
\[ G(X) \leq B, X \geq 0 \] are standard restrictions, \( g_i(X) \leq b_i, i=1, M \) where \( b_i \) - a set of real numbers, and \( g_i(X) \) are assumed continuous and convex.

\[
S = \{X \in \mathbb{R}^n | X \geq 0, G(X) \leq B, X^{\min} \leq X \leq X^{\max} \neq \emptyset \ (5)\]

where the set of admissible points set by restrictions (3)-(4) is not empty and represents a compact.

The vector minimization criterion \( F_2(X) \) can be transformed to the vector maximization criterion by the multiplication of each component of \( F_2(X) \) to the minus unit. The vector criterion of \( F_2(X) \) is injected into the vector problem of mathematical programming (1)-(4) to show that, in a problem, there are two subsets of criteria of \( K_1, K_2 \) with, in essence, various directions of optimization.

We assume that the optimum points received by each criterion do not coincide for at least two criteria. If all points of an optimum coincide among themselves for all criteria, then we consider the decision trivially.

### 2.2 Research and Analysis of the Vector Optimization Problem

At the beginning of the 20th century, during research into commodity exchange, Pareto\(^1\) mathematically formulated the criterion of optimality, the purpose of which is to estimate whether the proposed change improves common welfare in an economy. Pareto’s criterion claims that any change which does not inflict loss on anyone and brings benefit to some is an improvement. The Pareto criterion was later transferred to optimization problems with a set of criteria, where problems were considered in which optimization meant improving one or more indicators (criteria), provided that others did not deteriorate. Multi-criteria optimization problems arose. As a rule, a set of criteria was represented as a vector of criteria (hence vector optimization problems or vector problems in mathematical programming (VPMP). I have proposed a vector problem of mathematical programming entry in the form (1)-(4). It was immediately clear that the set point of \( S^0 \) is commensurable or can even coincide with a set of admissible points of \( S \). Below are two examples (VPLP\(^5\)\(^\circ\)\(^\circ\)) illustrating this premise.

**Example 1:** \( \max \ f(X) = \{ \max f_1(X) = 2.0x_1 + x_2, \max f_2(X) = x_1 + 2.0x_2 \} \) \( x_1 + x_2 = 1.0, x_1 \geq 0, x_2 \geq 0 \)

**Example 2:** \( \max \ f_1(X) = \{ \max f_1(X) = 2.0x_1 + x_2, \max f_2(X) = x_1 + 2.0x_2 \}, \min f_2(X) = \{ \max f_3(X) = x_1 + x_2 \} \) \( x_1 + x_2 \leq 1.0, x_1 \geq 0, x_2 \geq 0 \)

The geometric interpretation of the distribution of the analysis and solution results is shown in Figure 1. Figure 1 showed that the Pareto Set \( S^0 \) and the set of valid points \( S \) are equal to each other and represent the set of points lying on the straight line \( X_1X_2 \) (Example 1) or in the region \( X_3X_2X_3 \) (Example 2).

![Figure 1](https://example.com/figure1.png)

**Figure 1. Geometrical interpretation of distribution point-sets in VPLP with two (A) and three criteria (B).**

The given examples showed that if Pareto-optimal points are found in VPLP, then only the admissible point is found, and no more. The answer to the question of what this is better than, other than points from a Pareto set, remains open.

For VPLP (1)-(4), there is a problem not only with the choice of a point, Pareto-optimal \( X^o \in S \), but also with definition, in that a point of \( X^o \in S^0 \subseteq S \) is "more optimum" than another point of \( X \in S, X \neq X^o \), i.e., the choice of the principle of optimality on a Pareto set.

Generally, we will try to formulate the problem of finding the solution to VPMP (1)-(4). According to us, the problem with finding the solution to VPMP is in the ability to solve three problems:
(1) First, select the point $X^0$ from the Pareto set $S^p \subset S$ and show its optimality with respect to other points belonging to the Pareto set, i.e., determine the center of symmetry;

(2) Second, for any point $X \in S^p \subset S$, show by which criterion $q \in K$ it (the point) is higher than the other criteria $k=1, \ldots, N$ and by how much;

(3) Third, the solution of the inverse problem, which consists in the following: if you know the limits of changing the priority criterion on the Pareto set $S^p$, (and this is easy to do when solving separately for each criterion), then under a given numerical value of the criterion, you should be able to find a point with an error not exceeding the specified one.

It is the solution of these three problems, both as a whole and its individual parts, that the efforts of most researchers of vector optimization have been directed\cite{2,21,23,28}.

For the last three decades, a large number of articles and monographs have been devoted to methods for solving vector (multi-criteria) tasks. These have detailed the theoretical research and methods in the following ways:

(1) Methods for solving vector problems of mathematical programming based on the folding of criteria based on weight coefficients;

(2) VLP solution-methods using criteria restrictions;

(3) Methods for solving vector problems of mathematical programming based on the target-programming;

(4) Methods for solving vector problems of mathematical programming based on searching for a compromise solution;

(5) Methods based on human-machine procedures for decision-making.

The research, analysis and shortcomings of the listed methods were first published more than thirty years ago in Mashunin's report\cite{3}, currently the analysis is presented in the fourth chapter\cite{21}. The analysis is carried out by comparing the results of the solution of the test case obtained by these methods with the method based on the normalization of criteria and the principle of guaranteed result\cite{4,19,21}, which is the basis of this work.

2.3 The Theory of Vector Optimization: The Axioms and the Principle of Optimality for Vector Problem of Mathematical Programming with the Equivalent Criteria

The theory of vector optimization includes axiomatic - theoretical foundations and methods for solving vector problems of mathematical programming with equivalent criteria and with a given criterion priority. The theory is a mathematical tool for modeling the "decision object", which allows you to select any point from the set of points that are optimal in Pareto, and show it as the center of symmetry. We presented axioms and methods for solving vector optimization problems (1)-(4) with equivalent criteria\cite{4,19}.

For simplicity of research, the criteria and restrictions of the VPMP (1)-(4) are represented by second degree polynomials, i.e., convex vector problems are considered, which also include vector linear programming problems. Convex VPMP are characterized by the property that an optimum point exists and there is only one such point (Weierstrass Theorem).

2.3.1 The Axioms for Vector Problem of Mathematical Programming with the Equivalent Criteria

Definition 1. Normalization of the criterion.

Normalizing criteria (mathematical operation: the shift plus rationing) presents a unique display of the function $f_k(X) \forall k \in K$, in a one-dimensional space of $R^l$ (the function $f_k(X) \forall k \in K$ represents a function of transformation from a $N$-dimensional Euclidean space of $R^N$ in $R^l$). To normalize criteria in vector problems, linear transformations will be used:

$$f_k(X) = a_k f_k(X) + c_k \forall k \in K \quad (5)$$

or

$$f_k(X) = (f_k(X) + c_k)/a_k \forall k \in K \quad (6)$$

Where $f_k(X), k = 1, \ldots, N$ - aged (before normalization) value of criterion; $f_k(X), k = 1, \ldots, N$ - the normalized value, $a_k, c_k$ - constants.

Normalization of criteria $f_k(X)=(f_k(X)+c_k)/a_k \forall k \in K$ is a simple (linear) invariant transformation of a polynomial, as a result of which the structure of the polynomial remains unchanged. In the optimization problem, the normalization of criteria $f_k(X)=(f_k(X)+c_k)/a_k \forall k \in K$ does not affect the result of the solution. Indeed, if the convex optimization problem is solved:

$$\max_{X \in S} f(X),$$

then at the optimum point $X^* \in S$: $df(X^*)/dx = 0$. 

In the general case (including the normalization of the criterion (6)), the problem is solved:

\[ \max_{x \in S} \left( a_k f'_k(X) + c_k \right), \]

then at the optimum point \( X^* \in S \):

\[ \frac{d(a_k f'_k(X^*) + c_k)}{dx} = a_k \frac{d(f'_k(X^*))}{dx} + \frac{dc_k}{dx} = 0. \]

The result is identical, i.e., the optimum point \( X_k^* \), \( k = \overline{1,K} \) is the same for non-normalized and normalized problems.

**Definition 2. Definition of the relative assessment of the criterion.**

In a vector problem of mathematical programming (1)-(4) we will enter designation:

\[ \lambda_k(X) = \frac{f_k(X) - f_k^0}{f_k^0 - f_k^0}, \quad \forall k \in K \] (7)

is the relative estimate of a point \( X \in S \) \( k \)th criterion \( f_k(X) \) - \( k \)th criterion at the point \( X \in S \); \( F_k^0 \) - value of the \( k \)th criterion at the point of optimum \( X_k \), obtained in vector problem (1)-(4) of individual \( k \)th criterion; \( f_k^0 \) is the worst value of the \( k \)th criterion (ant optimum) at the point \( X_k^0 \) (Superscript 0 - zero) on the admissible set \( S \) in vector problem of mathematical programming (1)-(4); the task at max (1), (3), (4) the value of \( f_k^0 \) is the lowest value of the \( k \)th criterion

\[ f_k^0 = \min_{x \in S} f_k(X) \forall k \in K_1 \] (8)

And problem min (2), (3), (4) the value of \( f_k^0 \)is the greatest value of the \( k \)th criterion:

\[ f_k^0 = \max_{x \in S} f_k(X) \forall k \in K_2 \] (9)

The relative estimate of the \( \lambda_k(X) \) \( \forall k \in K \) is first, measured in relative units; secondly, the relative assessment of the \( \lambda_k(X) \) \( \forall k \in K \) on the admissible set is changed from zero in a point of \( X_k^0 \):

\[ \forall k \in K \lim_{x \to X_k^0} \lambda_k(X) = 0 \]

to the unit at the point of an optimum of \( X_k^0 \):

\[ \forall k \in K \lim_{x \to X_k^0} \lambda_k(X) = 1 \]
\[ \forall k \in K \ 0 \leq \lambda_k(X) \leq 1, \ X \in S \] (10)

As a result of this normalization, all the criteria of the VPMP are (1)-(4) are comparable in relative units, which allows comparing them with each other, using criteria for joint optimization.

**Axiom 1. About equality and equivalence of criteria in an admissible point of VPMP.**

In of Vector Problems of Mathematical Programming two criteria with the indexes \( k \in K, q \in K \) shall be considered as equal in \( x \in S \) point if relative estimates on \( k \)th and \( q \)th to criterion are equal among themselves in this point, i.e., \( \lambda_k(X) = \lambda_q(X), k, q \in K \). We will consider criteria equivalent in vector problems of mathematical programming if in, \( x \in S \) point when comparing in the numerical size of relative estimates of \( \lambda_k(X), k = \overline{1,K}, \) among themselves, on each criterion of \( f_k(X), k = \overline{1,K}, \) and, respectively, relative estimates of \( \lambda_k(X), \) isn't imposed conditions about priorities of criteria.

**Definition 3. Definition of a minimum level among all relative estimates of criteria.**

The relative level \( \min \) in a vector problem of mathematical programming represents the lower assessment of a point of \( X \in S \) among all relative estimates of \( \lambda_k(X), k = \overline{1,K} \):

\[ \forall X \in S \lambda \leq \lambda_k(X), k = \overline{1,K} \] (11)

the lower level for performance of a condition (11) in an admissible point of \( X \in S \) is defined by a formula:

\[ \forall X \in S \lambda = \min_{x \in S} \lambda_k(X) \] (12)

Ratios (11) and (12) are interconnected. They serve as a transition from operation (12) of definition of min to restrictions (11) and vice versa.

The level \( \lambda \) allows to unite all criteria in a vector problem of mathematical programming one numerical characteristic of \( \lambda \) and to make over her certain operations, thereby, carrying out these operations over all criteria measured in relative units. The level \( \lambda \) functionally depends on the \( X \in S \) variable, changing \( X \), we can change the lower level \( \lambda \). From here we will formulate the rule of search for the optimum decision.

**2.3.2 The Principle of Optimality for Vector Problem of Mathematical Programming with the Equivalent Criteria**

**Definition 4. The principle of an optimality with equivalent criteria.**
The vector problem of mathematical programming at equivalent criteria is solved, if the point of $X^o \in S$ and a maximum level of $\lambda^o$ (the top index o - optimum) among all relative estimates such that is found: 

$$\lambda^o = \text{max}_{X \in S} \text{min}_{k \in K} \lambda_k(X) \ (13)$$

Using interrelation of expressions (11) and (12), we will transform a maximum problem (13) to an extreme problem:

$$\lambda^o = \text{max}_{X \in S} \lambda \ (14)$$

at restriction

$$\lambda \leq \lambda_k(X), \ k = \overline{1, K} \ (15)$$

The resulting problem of mathematical programming (14)-(15) let's call the $\lambda$-problem. $\lambda$-problem (14)-(15) has $(N+1)$ dimension, as a consequence of the result of the solution of $\lambda$-problem (14)-(15) represents an optimum vector of $X^o \in R^{N+1}, \ (N+1)$ which component an essence of the value of the $\lambda^o$, i.e., $X^o = \{x_1^o, x_2^o, \ldots, x_{N+1}^o \}$, thus $x_{N+1}^o = \lambda^o$, and $(N+1)$ a component of a vector of $\lambda^o$ selected in view of its specificity.

The received a pair of $\{\lambda^o, X^o\} = X^o$ characterizes the optimum solution of $\lambda$-problem (14)-(15) and according to vector problem of mathematical programming (1)-(4) with the equivalent criteria, solved on the basis of normalization of criteria and the principle of the guaranteed result. We will call in the optimum solution of $X^o = \{\lambda^o, X^o\}, X^o$- an optimal point, and $\lambda^o$ - a maximum level.

An important result of the algorithm for solving vector problem of mathematical programming (1)-(4) with equivalent criteria is the following theorem.

**Theorem 1. The theorem of the two contradictory criteria in the VPMP with equivalent criteria.**

In convex vector problems of mathematical programming (1)-(4) at the equivalent criteria which is solved on the basis of normalization of criteria and the principle of the guaranteed result, in an optimum point of $X^o = \{\lambda^o, X^o\}$ two criteria are always - denote their indexes $q \in K, p \in K$ (which in a sense are the most contradiction of the criteria $k = \overline{1, K}$), for which equality is carried out:

$$\lambda^o = \lambda_q(X^o) = \lambda_p(X^o), \ q, p \in K, X^o \in S \ (16)$$

other criteria are defined by inequalities:

$$\lambda^o \leq \lambda_q(X^o), \ \forall q \in K, q \neq p \ (17)$$

For the first time, the proof of Theorem 1 is presented in the research of Mashunin and Levitskii\[6\], and later it is repeated in Mashunin’s study\[9\], but it is not really clear until now.

Along with the fact that the point $X^o$ is the optimal solution of the VPMP, the point $X^o$ is also the center of symmetry for at least two of the most contradictory criteria in the VPMP (1)-(4). We will show the study of symmetry in the next section, both for vector problems of linear and convex programming.

### 2.4 Mathematical Method of the Solution of Vector Problem with the Equivalent Criteria

To solve the vector problem of mathematical programming (1)-(4) the methods based on axiomaticity of the normalization of criteria and the principle of the guaranteed result, which follow from Axiom 1 and the principle of optimality 1, formulated in Section 2.3\[5,18\].

We will present in the form of a number of steps:

**Step 1. The solution of vector problem with mathematical programming.**

For $\forall k \in K_1$ is solved at the maximum, and for $\forall k \in K_2$ is solved at a minimum. $X^*_k$ - an optimum point by the corresponding criterion, $k = \overline{1, K}$; $f^*_k = f_k(X^*_k)$ - the criterion size kth in this point, $k = \overline{1, K}$.

**Step 2. The worst value of each criterion on S:** $f^*_k, k = \overline{1, K}$. For what the problem (1)-(4) for each criterion of $k = \overline{1, K_1}$ on a minimum is solved.

$$f^*_k = \min f_k(X), \ G(X) \leq B, X \geq 0, \ k = \overline{1, K_1} \ (14)$$

The VPMP (1)-(4) for each criterion $k = \overline{1, K_2}$ maximum is solved:

$$f^0_k = \max f_k(X), \ G(X) \leq B, X \geq 0, \ k = \overline{1, K_2} \ (15)$$
As a result of the decision we will receive: \( X^0_k = \{x_{ij} = 1_i, \bar{N}_j \} \) - an optimum point by the corresponding criterion, \( k = \bar{1}, \bar{K} \); \( f_k^0 = f_k(X^0_k) \) - the criterion size \( k \)th a point, \( X^0_k, k = \bar{1}, \bar{K} \).

Step 3. The system analysis of a set of points, optimum across Pareto, for this purpose in optimum points of \( X^* = \{X_k, k = \bar{1}, \bar{K} \} \), are defined sizes of criterion functions of \( F(X^*) \) and relative estimates \( \lambda (X^*) \).

\[
\lambda_k(X) = \frac{f_k(X) - f_k^0}{f_k - f_k^0}, \quad \forall k \in K 
\]

\[
F(X^*) = \{f_k(X_k^*), q = \bar{1}, \bar{K}, k = \bar{1}, \bar{K}\} = \begin{bmatrix}
f_1(X_1^\ast), & ... & f_K(X_K^\ast) \\
\end{bmatrix} 
\]

\[
\lambda(X^*) = \{\lambda_k(X_k^*), q = \bar{1}, \bar{K}, k = \bar{1}, \bar{K}\} = \begin{bmatrix}
\lambda_1(X_1^\ast), & ... & \lambda_K(X_K^\ast) \\
\end{bmatrix} 
\]

As a whole on a problem \( \forall k \in K \) the relative assessment of \( \lambda_k(X), k = \bar{1}, \bar{K} \) lies within \( 0 \leq \lambda_k(X) \leq 1, k = \bar{1}, \bar{K} \).

Step 4. Creation of the \( \lambda \)-problem.

Creation of a \( \lambda \)-problem is carried out in two stages: initially built the maximum problem of optimization with the normalized criteria which at the second stage will be transformed to the standard problem of mathematical programming called \( \lambda \)-problem.

For construction maximize a problem of optimization we use definition 2 - relative level:

\[
\forall X \in S \lambda = \min_{k \in K} \lambda_k(X) \quad (18)
\]

The bottom \( \lambda \) level is maximized on \( X \in S \), as a result we will receive a maximum problem of optimization with the normalized criteria:

\[
\lambda^o = \max_{X \in S} \min_{k \in K} \lambda_k(X) \quad (13)
\]

At the second stage we will transform a problem (18) to a standard problem of mathematical programming:

\[
\lambda^o = \max_{X \in S} \lambda, \rightarrow \lambda^o = \max_{X \in S} \lambda \quad (19)
\]

\[
\lambda - \lambda_k(X) \leq 0, k = \bar{1}, \bar{K} \rightarrow \lambda - \frac{f_k(X) - f_k^0}{f_k - f_k^0} \leq 0, k = \bar{1}, \bar{K} \quad (20)
\]

\[
G(X) \leq B, X \geq 0, \rightarrow G(X) \leq B, X \geq 0 \quad (21)
\]

Where the unknown vector of \( X \) has dimension of \( N+1 \): \( X = \{\lambda, x_1, ..., x_N\} \).

Step 5. Solution of \( \lambda \)-problem.

\( \lambda \)-problem (19)-(21) is a standard problem of convex programming and for its decision standard methods are used.

As a result of the solution of \( \lambda \)-problem it is received: \( X^o = \{X^o, \lambda^o\} \) - an optimum point;

\( f_k(X^o), k = \bar{1}, \bar{K} \) are values of the criteria in this point; \( \lambda_k(X^o) = \frac{f_k(X) - f_k^0}{f_k - f_k^0}, k = \bar{1}, \bar{K} \) are sizes of relative estimates; \( \lambda^o \) is the maximum relative estimate which is the maximum bottom level for all relative estimates of \( \lambda_k(X^o) \), or the guaranteed result in relative units. \( \lambda^o \) guarantees that all relative estimates of \( \lambda_k(X^o) \) more or are equal \( \lambda^o \):

\[
\lambda^o \leq \lambda_k(X^o), k = \bar{1}, \bar{K}, X^o \in S \quad (22)
\]

and according to the theorem \([5,14]\) point of \( X^o = \{\lambda^o, x_1, ..., x_N\} \) is optimum across Pareto.

2.5. Digital Transformation of the Vector Optimization Problem Solution

2.5.1 Solving a VPLP

Example 3. We consider a vector linear programming problem with two homogeneous criteria and two variables.

\[
\begin{align*}
\text{max } F_1(X) &= \{\text{max } f_1(X) = 10x_1 + 2x_2, \quad (23) \\
\text{max } f_2(X) &= 30x_1 + 180x_2 \} \quad (24)
\end{align*}
\]

\[
4x_1 + 5x_2 \leq 240 \quad (25)
\]
A method based on the principle of optimality of a vector problem with equivalent criteria is used to solve VPLP (23)-(26). Method for the solution of VPMP is presented in Section 2.4. We will show the solution to a VPLP in the MATLAB system.

A vector target function (23)-(26) in the form of a matrix is formed:

disp ('solution to a vector problem in linear programming')

cvec = 
\begin{bmatrix}
10 & 20 \\
30 & 180
\end{bmatrix}

The matrix of linear restrictions was

\[ a = \begin{bmatrix} 4 & 5 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \]

b = [240.0 56.0 40.0]% of the vector containing restrictions (bi).

Aeq = []; beq = [] % of restrictions such as equality.

\( X_0 = [0.0 \ 0.0] \); % of a vector of variables.

The algorithm to solve a vector problem in linear programming (23)-(26) is represented as a sequence of steps.

**Step 1. Decision for each criterion.**

The decision for the first criterion:

\[ [x_1, f_1] = \text{linprog}(cvec(1,:), a, b, Aeq, beq) \]

Decision for the first criterion:

\( X_1^* = \{ x_1 = 56, x_2 = 3.2 \} \)

\( f_1^* = -556.4 \)

The decision for the second criterion:

\( X_2^* = \{ x_1 = 10, x_2 = 40 \} \)

\( f_2^* = -7500 \)

The geometric interpretation of constraints, analysis results, and solutions is shown in Figure 2A.

**Figure 2. Geometric interpretation of the constraints](A) and results (B) of the VPLP solution \([\lambda_1(X), \lambda_2(X)]\).**

Notes: optimum points \( X^o \), relative estimates: \( \lambda^o \).
Step 2. The worst optimum point is determined by each criterion (anti-optimum) by multiplying the criterion by the minus unit. 

Decision for the first (23) and second (24) criteria:

\[
X^0_1 = \{x_1 = 0, x_2 = 0\}; \quad f^0_1 = f_{\text{max}} = 0 \quad (27)
\]

\[
X^0_2 = \{x_1 = 0, x_2 = 0\}; \quad f^0_2 = f_{\text{min}} = 0 \quad (28)
\]

Step 3. A system analysis of the criteria for vector problems in linear programming (23)-(26) is made, i.e., the system of two criteria at optimum points is analyzed.

For this purpose, optimum points of \(X_1^1\) and \(X_2^1\) are defined sizes of criterion functions and relative estimates of:

\[
F(X^*) = \|f_q(X^*)\|_{k=1}^{R} \quad (29)
\]

\[
\lambda(X^*) = \|\lambda_q(X^*)\|_{k=1}^{R} \quad (30)
\]

Step 4. The \(\lambda\)-problem is under construction.

Substituting numerical data, we obtain: \(\lambda^0 = \max_{x \in S} \lambda\).

with restrictions:

\[
\begin{cases}
\lambda - \frac{f_1(X^*)}{f_1^0} \leq 0 \\
\lambda - \frac{f_2(X^*)}{f_2^0} \leq 0
\end{cases} \quad (31)
\]

\[
\begin{cases}
\lambda - (10x_1 + 20x_2)/f_1^0 \leq 0 \\
\lambda - (30x_1 + 180x_2)/f_1^0 \leq 0 \\
4x_1 + 5x_2 \leq 240 \\
0 \leq x_1 \leq 56 \\
0 \leq x_2 \leq 40
\end{cases} \quad (32)
\]

Step 5. Solving the \(\lambda\)-problem.

Results of solving the \(\lambda\)-problem: optimum values of variables: \(X^0 = \{x_1 = 32.7, x_2 = 21.8\}\), \(\lambda^0 = 0.6567\); optimum value of criterion function \(\lambda^0 = 0.6567\).

We will execute a check at an optimum point of \(X^0 = \{x_1 = 32.7, x_2 = 21.8\}\) and determine the sizes of criterion functions of \(f_k^0 = \{f_k(X^0), k = 1, R\}\) and relative estimates of \(\lambda^0_k = \{\lambda^0_k(X^0), k = 1, R\}\). As a result, we will obtain: \(f_1(X^0) = -370.8, f_2(X^0) = -370.8, \lambda_1(X^0) = 0.6567, \lambda_2(X^0) = 0.6567\). This result confirms the proofs of Theorem 1, i.e., \(\lambda^0 = \lambda_1(X^0) = \lambda_2(X^0), 1, 2 \in K, X^0 \in S\).

Theoretically, the point \(X^0\) is the center of symmetry, and this is shown in Figure 2B in a three-dimensional space: \(x_1, x_2, \lambda\), where the normalized criteria are represented: \(\lambda_1(X), \lambda_2(X)\).

2.5.2 Solving a Vector Problem of Nonlinear Programming

Example 4. The consideration of the vector nonlinear (convex) programming problem with four homogeneous criteria. In terms of criteria, we use a circle, and with the linear restrictions the vector problem is therefore solved orally and imposed on variables.

\[
\text{opt } F(X) = \{\min F_2(X) = \min f_4(X) = (x_1 - 2)^2 + (x_2 - 2)^2\} \quad (33)
\]

\[
\min f_2(X) = (x_1 - 2.0)^2 + (x_2 + 1)^2 \quad (34)
\]

\[
\min f_3(X) = (x_1 + 1)^2 + (x_2 + 1)^2 \quad (35)
\]

at restrictions,

\[
\begin{cases}
0 \leq x_1 \leq 1 \\
0 \leq x_2 \leq 1
\end{cases} \quad (36)
\]

A geometrical interpretation of this VPMP (32)-(35) was presented in Figure 4.
To solve the problem (33)-(36) for each criterion, as well as the further \( \lambda \)-problem, the MATLAB system (the function \( \text{fmincon} (...) \) - the solution to a non-linear problem of optimization) is used\(^{24}\).

**Step 1. Decision for each criterion.**
The results of the decision vector problem of mathematical programming (33)-(36) in each criterion in the field of restrictions (36) are presented in Figure 3A at salient points:

\[
\begin{align*}
X_1^1 &= \{x_1 = 1, x_2 = 1\} \\
X_2^1 &= \{x_1 = 1, x_2 = 0\} \\
X_3^1 &= \{x_1 = 0, x_2 = 0\} \quad (37) \\
X_4^1 &= \{x_1 = 0, x_2 = 1\} \\
f_1^1 &= f_2^1 = f_3^1 = f_4^1 = 2
\end{align*}
\]

**Step 2. The worst optimum point is determined by each criterion (anti-optimum) by multiplying the criterion by the minus unit.**

\[
\begin{align*}
X_1^0 &= \{x_1 = 0, x_2 = 0\} \\
X_2^0 &= \{x_1 = 0, x_2 = 1\} \\
X_3^0 &= \{x_1 = 1, x_2 = 1\} \quad (38) \\
X_4^0 &= \{x_1 = 1, x_2 = 0\} \\
f_1^0 &= f_2^0 = f_3^0 = f_4^0 = 8
\end{align*}
\]

**Step 3. A system analysis of the criteria for vector problems in linear programming (33)-(36) is made, i.e., the system of four criteria at optimum points is analyzed.**

For this purpose, optimum points of \( X_1^1, X_2^1, X_3^1 \) and \( X_2^0 \) are defined sizes of criterion functions and relative estimates of:

\[
F(X^*) = \left\| f_q(X^*_k) \right\|_{q = 1, K}^{k = 1, K} \quad (29)
\]

\[
\lambda(X^*) = \left\| \lambda_q(X^*_k) \right\|_{q = 1, K}^{k = 1, K} \quad (30)
\]

The Pareto set lies between optimum points \( X_1^1, X_2^1, X_3^1, X_4^1 \), i.e., the area of admissible points of \( S \) formed by restrictions (36) coincides with a point set, which is Pareto-optimal \( S^P \), \( S^P = S \).

At points of an optimum of \( X_k^1, k = 1, K \) all relative estimates (the normalized criteria) are equal to the unit:

\[
\lambda_k(X_k^1) = \frac{f_k(X_k^1) - f_k^0}{f_k^0} = 1, \quad k = 1, K, K = 4 \quad (7)
\]

At points of an optimum of \( X_k^0, k = 1, K \) (anti optimum) all relative estimates are equal to zero:
\[ \lambda_k(X_k^0) = \frac{f_k(X_k^0) - f_k^0}{f_1^0 - f_k^0} = 0, \ k = 1, K, K = 4. \] 
From here, \( \forall k \in K, \forall X \in S, 0 \leq \lambda(X) \leq 1. \)

**Step 4. The \( \lambda \)-problem is under construction.**

\[ \lambda_{o^*} = \max \lambda \quad (19) \]

with restrictions,

\[
\begin{align*}
\lambda - \frac{f_1(X^o) - f_1^0}{f_1^0 - f_1^0} & \leq 0 \\
\lambda - \frac{f_2(X^o) - f_2^0}{f_2^0 - f_2^0} & \leq 0 \\
\lambda - \frac{f_3(X^o) - f_3^0}{f_3^0 - f_3^0} & \leq 0 \\
\lambda - \frac{f_4(X^o) - f_4^0}{f_4^0 - f_4^0} & \leq 0 \\
0 & \leq x_1 \leq 1, \ 0 \leq x_2 \leq 1
\end{align*}
\]

**Step 5. Solving the \( \lambda \)-problem.**

The optimum values of variables:

\[ X^o = \{ x_1 = 0.5, x_2 = 0.5 \} \]

\[ \lambda_{o^*} = 0.5833 \]

As in a task (23)-(26) criteria are symmetric and the point of the guaranteed result of \( X^o = \{ x_1 = 0.5, x_2 = 0.5 \} \) is easily defined - it lies in the center of a square, and a maximal relative assessment of \( \lambda_{o^*} = 0.5833 \). Really, for example, with the first criterion:

\[ \lambda_1(X^o) = \frac{f_1(X^o) - f_1^0}{f_1^0 - f_1^0} = ((0.5 - 2)^2 + (0.5 - 2)^2 - 8) / (2 - 8) = 0.5833 \]

And similarly for other criteria: \( \lambda_{o^*} = \lambda_2(X^o) = \lambda_3(X^o) = \lambda_4(X^o) = 0.5833 \).

This was very clearly shown in Figure 3B.

In Figure 3A and 3B we can see that the area (point set) limited to points of \( S_q = \{ X_1 = X_{1_{opt}}X_{12}X^oX_{41} \} \) is characterized by \( \lambda_1(X) \geq \lambda_k(X), k = 2, \ldots, 4 \), Figure 3A showed how \( \lambda_1(\geq \lambda_2, \lambda_3, \lambda_4) \), i.e., it was prioritized by the first criterion. In this area, the priority of the first criterion is always more or equal to the unit: \( p_k^1(X) = \lambda_1(X) / \lambda_k(X) \geq 1, \forall X \in S_1 \).

Areas (point set) prioritized by the corresponding criterion are similarly shown. In total, they give a point set, which is Pareto-optimal, of \( S^o \), and it (for this example) is equal to a set of admissible points: \( S^o = S_1^o \cup S_2^o \cup S_3^o \cup S_4^o \cup X^o = S \).

To solve a problem (25)-(29) with two criteria, for example, the third and fourth, and then the point set, are Pareto-optimal, and lie on a piece \( X_3X_4 \), and the \( X^o \) point defines the result of the decision \( \lambda_{oo} \) - the maximum level, and \( \lambda_{oo} = \lambda_3(X^o) = \lambda_4(X^o) = 0.791 \) according to Theorem 1. At the same time, both numerical and geometric symmetry is preserved.

Thus, vector optimization is a mathematical tool for studying in various systems.

### 2.6 The Theory of Vector Optimization: The Axioms and the Principle of Optimality for Vector Problem of Mathematical Programming with a Criterion Priority

For development of methods of the solution of vector problems of mathematical programming with a priority of criterion we use definitions as follows: Priority of one criterion of VPMP, with a criterion priority over other criteria; Numerical expression of a priority; The set priority of a criterion; the lower (minimum) level from all criteria with a priority of one of them; a subset of points with priority by criterion (Axiom 2); the principle of optimality of the solution of problems of vector optimization with the set priority of one of the criteria, and related theorems. For more details see\(^{[19,21]}\).

#### 2.6.1 The Axioms for Vector Problem of Mathematical Programming with the Criterion Priority

**Definition 4.** About the priority of one criterion over the other in the VPMP.
The criterion of \( q \in K \) in the vector problems of mathematical programming of Equations (1)-(4) in a point of \( X \in S \) has priority over other criteria of \( k = \overline{1}, \overline{R} \), and the relative estimate of \( \lambda_q(X) \) by this criterion is greater than or equal to relative estimates of \( \lambda_d(X) \) of other criteria:
\[
\lambda_q(X) \geq \lambda_k(X), k = \overline{1}, \overline{R} \quad (30)
\]
And a strict priority for at least one criterion of \( t \in K \), \( \lambda_q(X) > \lambda_d(X) \), \( t \neq q \), and for other criteria of \( \lambda_q(X) \geq \lambda_k(X), k = \overline{1}, \overline{R}, k \neq t \neq q \)

Introduction of the definition of a priority of criterion \( q \in K \) in the VPMP of Equations (1)-(4) executed the redefinition of the early concept of a priority. Earlier the intuitive concept of the importance of this criterion was outlined, now this “importance” is defined as a mathematical concept: the higher the relative estimate of the \( q \)th criterion compared to others, the more it is important (i.e., more priority), and the highest priority at a point of an optimum is \( X_q^*, \forall q \in K \). From the definition of a priority of criterion of \( q \in K \) in the VPMP of Equations (1)-(4), it follows that it is possible to reveal a set of points \( S_q \subset S \) that is characterized by \( \lambda_q(X) \geq \lambda_d(X), \forall k \neq q, \forall X \in S_q \). However, the answer to whether a criterion of \( q \in K \) at a point of the set \( S_q \) has more priority than others does remain open. For clarification of this question, we define a communication coefficient between a couple of relative estimates of \( q \) and \( k \) that, in total, represent a vector: \( P^q(X) = \{ p_q^k(X) | k = \overline{1}, \overline{R}, q \in K \forall X \in S_q \} \).

**Definition 5. About numerical expression of a priority of one criterion over another in the VPMP.**

In the vector problems of mathematical programming of Equations (2) and (3), with priority of the \( q \)th criterion over other criteria of \( k = \overline{1}, \overline{R} \), for \( \forall X \in S \), and a vector of \( P^q(X) \) which shows how many times a relative estimate of \( \lambda_d(X), q \in K \), is more than other relative estimates of \( \lambda_k(X), k = \overline{1}, \overline{R} \), we define a numerical expression of the priority of the \( q \)th criterion over other criteria of \( k = \overline{1}, \overline{R} \) as:
\[
P^q(X) = \{ p_q^k(X) = \frac{\lambda_q(X)}{\lambda_k(X)}, k = \overline{1}, \overline{R} \} \quad (40)
\]
\[
p_q^k(X) \geq 1, \forall X \in S_q \subset S, k = \overline{1}, \overline{R}, \forall q \in K \quad (41)
\]

**Definition 6. About the set numerical expression of a priority of one criterion over another in the VPMP.**

In the vector problems of mathematical programming of Equations (1)-(4) with a priority of criterion of \( q \in K \) for \( \forall X \in S \), vector \( P^q = \{ p_q^k, k = \overline{1}, \overline{R} \} \) is considered to be set by the person making decisions (i.e., decision-maker) if everyone is set a component of this vector. Set by the decision-maker, component \( p_q^k \), from the point of view of the decision-maker, shows how many times a relative estimate of \( \lambda_d(X), q \in K \), is greater than other relative estimates of \( \lambda_k(X), k = \overline{1}, \overline{R} \). The vector of \( p_q^k, k = \overline{1}, \overline{R} \), is the numerical expression of the priority of the \( q \)th criterion over other criteria of \( k = \overline{1}, \overline{R} \):
\[
P^q(X) = \{ p_q^k(X), k = \overline{1}, \overline{R} \} \quad (40)
\]
\[
p_q^k(X) \geq 1, \forall X \in S_q \subset S, k = \overline{1}, \overline{R}, \forall q \in K \quad (41)
\]

The vector problems of mathematical programming of Equations (1)-(4), in which the priority of any criteria is set, is called a vector problem with the set priority of criterion. In the comparison of relative estimates with a priority of criterion of \( q \in K \) we define the additional numerical characteristic of \( \lambda \) which we call the level.

**Definition 7. About the lower level among all relative estimates with a criterion priority in the VPMP.**

The \( \lambda \) level is the lowest among all relative estimates with a priority of criterion of \( q \in K \) such that:
\[
\lambda \leq p_q^k \lambda_k(X), k = \overline{1}, \overline{R}, q \in K, \forall X \in S_q \subset S \quad (42)
\]
The lower level for the performance of the condition in Equation (18) is defined as:
\[
\lambda = \min_{k \in K} p_q^k \lambda_k(X), q \in K, \forall X \in S_q \subset S \quad (18)
\]

Equations (42) and (18) are interconnected and serve as a further transition from the operation of the definition of the minimum to restrictions, and vice versa. In Section 2.3, we gave the definition of a Pareto optimal point \( X^0 \in S \) with equivalent criteria. Considering this definition as an initial one, we will construct a number of the axioms dividing an admissible set of \( S \) into, first, a subset of Pareto optimal points \( S^0 \), and, secondly, a subset of points \( S_q \subset S, q \in K \), with priority for the \( q \)th criterion.
Axion 2. About a subset of points, priority by criterion in the VPMP.
In the vector problems of mathematical programming of Equations (1)-(4), the subset of points \( S_q \subseteq S \) is called the area of priority of criterion of \( q \in K \) over other criteria, if
\[
\forall X \in S_q, \forall k \in K, \lambda_q(X) \geq \lambda_q(X), q \neq k \quad (43)
\]
This definition extends to a set of Pareto optimal points \( S^o \) that is given by the following definition.

Axion 2a. About a subset of points, priority by criterion, on Pareto's great number in a vector problem.
In a vector problem of mathematical programming the subset of points \( S_q^o \subset S \) is called the area of priority of criterion of \( q \in K \) over other criteria, if \( \forall X \in S_q^o, \forall k \in K, \lambda_q(X) \geq \lambda_q(X), q \neq k \). In the following we provide explanations.

Axion 2 and 2a allow the breaking of the vector problems of mathematical programming in Equations (1)-(4) into an admissible set of points \( S \), including a subset of Pareto optimal points, \( S^o \subset S \), and subsets:
One subset of points \( S \subseteq S \) where criteria are equivalent, and a subset of points \( S^o \) crossed with a subset of points \( S^o \), allocated to a subset of Pareto optimal points at equivalent criteria \( S^o = S \cap S^o \). As will be shown further, this consists of one point of \( S^o \subseteq S \), i.e., \( S^o = S \cap S^o \).

"\( K \)" subsets of points where each criterion of \( q \in K \) has a priority over other criteria of \( k = 1, K \), \( q \neq k \), and thus breaks, first, sets of all admissible points \( S \), into subsets \( S_q \subseteq S \), \( q = 1, K \) and, second, a set of Pareto optimal points, \( S^o \), into subsets \( S_q^o \subseteq S^o \), \( q = 1, K \). This yields:
\[
(S \cup (U_{q \in K} S_q^o)) = S^o, S_q^o \subset S^o \subset S, q = 1, K \quad (44)
\]
We note that the subset of points \( S_q^o \), on the one hand, is included in the area (a subset of points) of priority of criterion of \( q \in K \) over other criteria: \( S_q^o \subset S_q \subset S \), and, on the other, in a subset of Pareto optimal points \( S_q^o \subset S^o \subset S \).

Axion 2 and the numerical expression of priority of criterion in the Definition 5 allow the identification of each admissible point of \( X \in S \) (by means of vector):
\[
P^q(X) = \{p^q_k(X) = \frac{\lambda_q(X)}{k_{\lambda = \lambda_q(X)}} \} k = 1, K \quad (45)
\]
A subset of points by priority criterion \( S_q^o \), which is included in a set of points \( S \), \( \forall q \in K X \in S \), (such a subset of points can be used in problems of clustering, but is beyond this article); a subset of points by priority criterion \( S_q^o \), which is included in a set of Pareto optimal points \( S^o \), \( \forall q \in K X \in S_q^o \subset S^o \). Thus, full identification of all points in the vector problem of Equations (1)-(4) is executed in sequence as:

Set of admissible points of \( X \in S \) → Subset of points, optimum across Pareto, \( X \in S^o \subset S \) → Subset of points, optimum across Pareto \( X S_q^o \subset S^o \subset S \) → Separate point of a \( \forall X \in S \) \( X S_q^o \subset S^o \subset S \)

This is the most important result which allows the output of the principle of optimality and to construct methods of a choice of any point of Pareto's great number.

2.6.2 The Principle of Optimality for Vector Problem of Mathematical Programming with the Criterion Priority
Definition 8. Principle of optimality 2. The solution of a vector problem with the set criterion priority in the VPMP.
The vector problem of Equations (1)-(4) with the set priority of the \( q \)th criterion of \( p^q_k \lambda_k(X), k = 1, K \), \( \lambda = 1, K \) is considered solved if the point \( X^o \) and maximum level \( \lambda^o \) among all relative estimates is found such that:
\[
\lambda^o = \max_{X^o \in S} \min_{k \in K} p^q_k \lambda_k(X), q \in K \quad (46)
\]
Using the interrelation of Equations (7) and (8), we can transform the maximum problem of Equation (46) into an extreme problem of the form:
\[
\lambda^o = \max_{X^o \in S} \hat{\lambda} \quad (14)
\]
at restriction
\[ \lambda \leq p^q_k \lambda_k(X), \quad k = 1, R \] (47)

We call Equations (36) and (37) the \( \lambda \)-problem with a priority of the \( q \)th criterion. The solution of the \( \lambda \)-problem is the point \( X^o = \{ X^o, \lambda^o \} \). This is also the result of the solution of the vector problem of mathematical programming of Equations (1)-(4) with the set priority of the criterion, solved on the basis of normalization of criteria and the principle of the guaranteed result. In the optimum solution \( X^o = \{ X^o, \lambda^o \}, X^o \), an optimum point, and \( \lambda^o \), the maximum bottom level, the point of \( X^o \) and the \( \lambda^o \) level correspond to restrictions of Equation (4), which can be written as:

\[ \lambda^{o} \leq p^q_k \lambda_k(X^o), \quad k = 1, R. \]

These restrictions are the basis of an assessment of the correctness of the results of a decision in practical vector problems of optimization. From Definitions 1 and 2, "Principles of optimality", follows the opportunity to formulate the concept of the operation "opt".

Definition 9. Mathematical operation "opt" in the VPMP.
In the vector problems of mathematical programming of Equations (1)-(4), in which "max" and "min" are part of the criteria, the mathematical operation "opt" consists of the definition of a point \( X^o \) and the maximum \( \lambda^o \) bottom level to which all criteria measured in relative units are lifted:

\[ \lambda^{o} \leq \lambda_k(X^o) = \frac{f_k(X)-f_k^o}{f_k^o-f_k^o}, \quad k = 1, R \] (48)
i.e., all criteria of \( \lambda_k(X^o), \quad k = 1, R \), are equal to or greater than the maximum level of \( \lambda^o \) (therefore \( \lambda^o \) is also called the guaranteed result).

Theorem 2. The theorem of the most inconsistent criteria in a VPMP with the set priority.
If in the convex VPMP of Equations (1)-(4) the priority of the \( q \)th criterion of \( p^q_k, k = 1, R, \forall q \in K \), other criteria is set, at a point of an optimum \( X^o \in S \) obtained on the basis of normalization of criteria and the principle of guaranteed result, there will always be two criteria with the indexes \( \lambda^o \in S, \quad r \in K, \quad t \in K \), for which the following strict equality holds:

\[ \lambda^{o} = p^r_k \lambda_r(X^o) = p^t_k \lambda_t(X^o), \quad r, t, \in K \] (49)

and other criteria are defined by inequalities:

\[ \lambda^{o} \leq p^q_k (X^o), \quad k = 1, R, \forall q \in K, \quad q \neq r \neq t \] (50)

Criteria with the indexes \( r \in K, \quad t \in K \), for which the equality of Equation holds are called the most inconsistent.

Proof. Similar to Theorem 2\textsuperscript{[10]}. We note that in Equations \( \lambda^o = p^q_k \lambda_r(X^o) = p^q_k \lambda_t(X^o) \), the indexes of criteria \( r, t \in K \) can coincide with the \( q \in K \) index.

Consequence of Theorem 1, about equality of an optimum level and relative estimates in a vector problem with two criteria with a priority of one of them.

In a convex VPMP with two equivalent criteria, solved on the basis of normalization of criteria and the principle of the guaranteed result, at an optimum point \( X^o \) equality is always carried out at a priority of the first criterion over the second:

\[ \lambda^{o} = \lambda_1(X^o) = p^1_2(X^o)\lambda_2(X^o), \quad X^o \in S \] (51)

where

\[ p^1_2(X^o) = \lambda_1(X^o)/\lambda_2(X^o) \]

and at a priority of the second criterion over the first:
\[
\lambda^o \leq \lambda_k(X^o), k = \overline{1,K}, X^o \in S \quad (22)
\]

The person making the decision carries out the analysis of the results of the solution of the vector problem with equivalent criteria. If the received results satisfy the decision maker, then the process concludes, otherwise subsequent calculations are performed.

In addition, we calculate that in each point \( X^*_k, k = \overline{1,K} \), we determine sizes of all criteria of: \( q = \overline{1,K}, (f^o_q(X^*_k), q = \overline{1,K} \), and relative estimates \( \lambda(X^*) = \{ \lambda_q(X^*_k), q = \overline{1,K}, k = \overline{1,K} \}, \lambda_k(X) = \frac{f_k(X^*) - f^o_k}{f^o_k}, \forall k \in K \):

\[
F(X^*) = \begin{bmatrix}
    f_1(X^*_1) & \cdots & f_K(X^*_1) \\
    \vdots & \ddots & \vdots \\
    f_1(X^*_K) & \cdots & f_K(X^*_K)
\end{bmatrix} \quad (53)
\]

\[
\lambda(X^*) = \begin{bmatrix}
    \lambda_1(X^*_1) & \cdots & \lambda_K(X^*_1) \\
    \vdots & \ddots & \vdots \\
    \lambda_1(X^*_K) & \cdots & \lambda_K(X^*_K)
\end{bmatrix} \quad (54)
\]

Matrices of criteria of \( F(X^*) \) and relative estimates of \( \lambda(X^*) \) show the sizes of each criterion of \( k = \overline{1,K} \) upon transition from one optimum point \( X^*_k, k \in K \) to another \( X^*_q, q \in K \), i.e., on the border of a great number of Pareto.

At an \( X^o \)-optimum point at equivalent criteria we calculate sizes of criteria and relative estimates:

\[
\begin{align*}
    f_k(X^o), k = \overline{1,K} \\
    \lambda_k(X^o), k = \overline{1,K}
\end{align*} \quad (55)
\]

In which satisfy the inequality of Equation \( \lambda^o \leq \lambda_k(X^o), k = \overline{1,K}, X^o \in S \). In other points \( X \in S^o \), in relative units the criteria of \( \lambda = \min_{k \in K} \lambda_k(X) \) are always less than \( \lambda^o \). This information is also a basis for further study of the structure of a great number of Pareto.

**Step 2. Choice of priority criterion of \( q \in K \).**

From theory (see Theorem 1) it is known that at an optimum point \( X^o \) there are always two most inconsistent criteria, \( q \in K \) and \( v \in K \), for which in relative units an exact equality holds: \( \lambda^o = \lambda_q(X^o) = \lambda_v(X^o), q, v \in K, X \in S \). Others are subject to inequalities: \( \lambda^o \leq \lambda_k(X^o), \forall k \in K, q \neq v \neq k \).

As a rule, the criterion which the decision-maker would like to improve is part of this couple, and such a criterion is called a priority criterion, which we designate \( q \in K \).

**Step 3. Numerical limits of the change of the size of a priority criterion \( q \in K \) are defined.**

For priority criterion \( q \in K \) from the matrix of Equation (53) and (54). The numerical limits of the change of the size of criterion: in physical units of \( f_q(X^o) \leq f_q(X^*) \leq f_q(X^o), k \in K \), where \( f_q(X^*_q) \) derives from the matrix of Equation

\[
\lambda^o = \lambda_2(X^o) = p^2_1(X^o)\lambda_1(X^o), X^o \in S \quad (52)
\]

Where

\[
p^2_1(X^o) = \lambda_2(X^o)/\lambda_1(X^o)
\]
$F(X^o)$, all criteria showing sizes measured in physical units, $f_k(x^o)$, $k = 1, \ldots, K$ from $\lambda_k(x^o)$, $k = 1, \ldots, K$, and, in relative units of:

$$\lambda_k(x^o) \leq \lambda_j(x^o) \leq \lambda_q(x^o), \quad k \in K \quad (56)$$

where $\lambda_q(x^o)$ derives from the matrix $\lambda(x^o)$, all criteria showing sizes measured in relative units (we note that $\lambda_q(x^o)=1, \lambda_q(x^o)$ from Equation (56).

Step 4. Choice of the size of priority criterion (decision-making).

The person making the decision carries out the analysis of the results of calculations of $F(X^o)$, $\lambda(x^o)$ and chooses the numerical size $f_q$ of the criterion of $q \in K$:

$$\lambda_q(x^o) \leq f_q \leq f_k(x^o), \quad q \in K \quad (57)$$

For the chosen size of the criterion of $f_q$ it is necessary to define a vector of unknown $X^o$. For this purpose, we carry out the subsequent calculations.

Step 5. Calculation of a relative assessment.

For the chosen size of the priority criterion of $f_q$ the relative assessment is calculated as:

$$\lambda_q = \frac{f_q - f_0}{f_0 - f_0} \quad (58)$$

which upon transition from point $X^o$ to $X_q$ according to Equation (58), lies in the limits: $\lambda_q(x^o) \leq \lambda_q \leq \lambda_q(x^o)=1$.

Step 6. Calculation of the coefficient of linear approximation.

Assuming a linear nature of the change of criterion of $f_q(x)$ and according to the relative assessment of $\lambda_q(x)$ in Equation (58), using standard methods of linear approximation we calculate the proportionality coefficient between $\lambda_q(x^o)$, $\lambda_q$, which we call $\rho$:

$$\rho = \frac{\lambda_q - \lambda_q(x^o)}{\lambda_q - \lambda_q(x^o)}, \quad q \in K \quad (59)$$

Step 7. Calculation of coordinates of priority criterion with the size $f_q$.

In accordance with Equation (58), the coordinates of the $X_q$ priority criterion point lie within the following limits: $X^o \leq X_q \leq X_q$, $q \in K$. Assuming a linear nature of change of the vector

$$X_q = \{x_q^1, \ldots, x_q^N\}$$

we determine coordinates of a point of priority criterion with the size $f_q$ with the relative assessment of $\lambda_q$:

$$X_q = \{x_q^1 = x_q^1 + \rho(x_q^1(1) - x_q^1) \} \quad (59)$$

where

$$X^o = \{x_q^1, \ldots, x_q^N\}$$

$$X_q^* = \{x_q^1(1), \ldots, x_q^N(N)\}$$

Step 8. Calculation of the main indicators of a point $x_q$.

For the obtained point $x_q$, we calculate all criteria in physical units:

$$F^q = \{f_k(x^q), k = 1, K\} \quad (60)$$

all relative estimates of criteria:

$$\lambda^q = \{\lambda^q_k, k = 1, K\} \quad (61)$$

$$\lambda_k(x^q) = \frac{f_k(x^q) - f_0}{f_0 - f_0}, \quad k = 1, K \quad (62)$$

the vector of priorities:

$$P^q = \{p^q_k = \frac{\lambda_k(x^q)}{\lambda_k(x^q)}, \quad k = 1, K\} \quad (63)$$

the maximum relative assessment:

$$\lambda^o q = \min \{p^q_k \lambda_k(x^q), k = 1, K\} \quad (64)$$
Any point from Pareto's set $X^a_i = \{x^a_i, x^{s}_i\} \in S^a$ can be similarly calculated.

The calculated size of criterion $f_q(X^a_i), q \in K$ is usually not equal to the set $f_q$. The error of the choice of $\Delta f_q = |f_q(X^a_i) - f_q|$ is defined by the error of linear approximation.

3 RESULTS AND DISCUSSION
3.1 Digital Transformation of Optimal Decision-making in Computer-aided Design of Economic and Engineering Systems

The process of the work, which has an applied nature, is aimed at illustrating the digital transformation of making optimal decisions in design automation: first, for economic systems: firms\([40-42]\), the market\([39,43]\), the region\([43,44]\) and the state as a whole\([44,45]\), in this work we will show the digital transformation of making optimal decisions on the example of a small firm; secondly, we will show the digital transformation of making optimal decisions on the example of engineering systems\([46-20]\).

3.1.1 Digital Transformation of Making Optimal Decisions in the Formation of the Production and Economic Plan of the Enterprise
3.1.1.1 Setting and Solution of the Production and Economic Plan of the Enterprise

Consider the construction of an enterprise’s production plan model, in which the vector variables are $N=4$.

What is given. The company produces four types of heterogeneous products. In the production of these, it uses three types of resources $M=3$: labor, material; power (the use of equipment and welding, turning, etc.). The symbols are similar to the previous problem. A technological matrix of production is presented in Table 1. This also indicates the enterprise’s potential for each type of resource $b_i, i = 1,3$, as well as income $c^1_j$ and profit $c^2_j$ from the sale of a unit of each type of product.

Table 1. The Consumption of Resources and Operational Performance

<table>
<thead>
<tr>
<th>Type of Resources</th>
<th>Costs of Resources per Product</th>
<th>Possibilities for the Firm Re. Resources</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Labor (people/week)</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Material (in kg.)</td>
<td>7</td>
<td>5</td>
</tr>
<tr>
<td>Capacity (per hour)</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>Income from a unit of production $c^1_j$</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>Profit $c^2_j$, $j=1,...,4$</td>
<td>2.0</td>
<td>10.0</td>
</tr>
<tr>
<td>Output</td>
<td>$x_1$</td>
<td>$x_2$</td>
</tr>
</tbody>
</table>

It is necessary to make a production schedule for the enterprise which includes indicators according to the products (of nomenclature) and on the basis of volume, i.e., how many products of the corresponding type should be made by the enterprise so that income and profit is possible, as above. We need to construct a mathematical model of the problem and solve it.

The construction of the problem is similar to the previous one. We will express the target orientation for the production schedule by the means of a vector problem in linear programming which will take the following form:

$$\text{opt } F(X) = \{\max f_1(X) = (4. x_1 + 5. x_2 + 9. x_3 + 11. x_4), \text{(65)}\}$$

$$\max f_2(X) = (2.0 x_1 + 10.0 x_2 + 6.0 x_3 + 20.0 x_4) \text{ (66)}$$

At restrictions,

$$x_1 + x_2 + x_3 + x_4 \leq 15$$

$$7.0 x_1 + 5.0 x_2 + 3.0 x_3 + 2.0 x_4 \leq 120$$

$$3.0 x_1 + 5.0 x_2 + 10.0 x_3 + 15.0 x_4 \leq 100$$

$$x_1 \geq 0.0, x_2 \geq 0.0, x_3 \geq 0.0, x_4 \geq 0.0$$

(67)
In this vector problem of mathematical programming, the following is formulated: it is necessary to find a solution to \( x_1, x_2, x_3, x_4 \) in a system of inequalities, at which the \( f_i(X) \) and \( f_j(X) \) functions accept perhaps maximum value.

### 3.1.1.2 Solving the VLP-Model of the Economic System

We will show the solution to a VLP in the MATLAB system according to an algorithm for solving vector problems in linear programming, on the basis of the normalization of criteria and the principle of a guaranteed result. Input data are prepared, the inclined font selected is the text of the program in the MATLAB system.

A vector target function in the form of a matrix is formed:

\[
\text{disp('solution to a vector problem in linear programming')}
\]

\[
cvec = [\begin{bmatrix} -4 & -5 & -9 & -11 \end{bmatrix} \ % \ of \ sales \ volume. \\
\begin{bmatrix} -2 & -10 & -6 & -20 \end{bmatrix} \ % \ of \ profit \ volume. \\
a = [\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}; \\
\begin{bmatrix} 7 & 5 & 3 & 2 \end{bmatrix}; \\
b = [\begin{bmatrix} 15.0 & 120.0 & 100.0 \end{bmatrix}] \ % \ of \ a \ matrix \ of \ linear \ restrictions. \\
\text{Aeq} = []; \ \text{beq} = [] \ % \ of \ restrictions \ such \ as \ equality. \\
\chi_0 = [\begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix}] \ % \ of \ a \ vector \ of \ variables.
\]

The algorithm to solve a vector problem in linear programming is represented as a sequence of steps.

#### Step 1. Decision for each criterion.

(1) The decision for the first criterion (65)

\[
[x_1, f_1] = \text{linprog(cvec, a, b, Aeq, beq, lb, ub)}
\]

Decision for the first criterion (65):

\[
X_1^* = x_1 \max = \{x_1 = 7.14, x_2 = 0, x_3 = 7.85, x_4 = 0\}
\]

\[
f_1^* = f_1 \max = -99.286
\]

(2) The decision for the second criterion (66)

\[
X_2^* = \{x_1 = 0, x_2 = 125, x_3 = 0, x_4 = 25\}
\]

\[
f_2^* = 175
\]

#### Step 2. The worst optimum point is determined by each criterion (anti-optimum) by multiplying the criterion by the minus unit.

Decision for the first (66) and second (66) criteria:

\[
X_1^0 = x_{\min} = \{x_1 = 0, \ldots, x_4 = 0\}
\]

\[
f_1^0 = f_{\max} = 0
\]

\[
X_2^0 = x_{\min} = \{x_1 = 0, \ldots, x_4 = 0\}
\]

\[
f_2^0 = f_{\min} = 0
\]

#### Step 3. A system analysis of the criteria for vector problems in linear programming is made, i.e., the system of two criteria at optimum points is analyzed.

For this purpose, optimum points of \( X_1^* \) and \( X_2^* \) are defined sizes of criterion functions and relative estimates of:

\[
F(X^*) = \|f_q(X^*)\|_{q = \Gamma_X}^{k = \Gamma_X} (29)
\]

\[
\lambda(X^*) = \|\lambda_q(X^*)\|_{q = \Gamma_X}^{k = \Gamma_X} (30)
\]

\[
X_1^* = \{x_1 = 7.14, x_2 = 0, x_3 = 7.85, x_4 = 0\}
\]

\[
F(X_1^*) = [\begin{bmatrix} f_1(X_1^*) & f_2(X_1^*) \end{bmatrix}] = [-99.29, 61.43]
\]

\[
\lambda(X_1^*) = [\begin{bmatrix} \lambda_1(X_1^*) & \lambda_2(X_1^*) \end{bmatrix}] = [1.0000, 0.3510]
\]

\[
X_2^* = \{x_1 = 0, x_2 = 125, x_3 = 0, x_4 = 25\}
\]

\[
F(X_2^*) = [\begin{bmatrix} f_1(X_2^*) & f_2(X_2^*) \end{bmatrix}] = [90.0, 175.0]
\]

\[
\lambda(X_2^*) = [\begin{bmatrix} \lambda_1(X_2^*) & \lambda_2(X_2^*) \end{bmatrix}] = [0.9065, 1.0000]
\]

#### Step 4. The \( \lambda \)-problem is under construction.

\[
\lambda^\circ = \max_{\lambda \in \mathbb{R}} \lambda (36)
\]
with restrictions.

\[
\begin{align*}
&\lambda - \frac{f_1(x)}{f_1^*} \leq 0 \\
&\lambda - \frac{f_2(x)}{f_2^*} \leq 0
\end{align*}
\] (31)

Substituting numerical data, we obtain (36) with restrictions:

\[
\begin{align*}
\lambda - \frac{4.0x_1 + 5.0x_2 + 9.0x_3 + 11.0x_4}{f_1^*} & \leq 0 \\
\lambda - \frac{2.0x_1 + 10.0x_2 + 6.0x_3 + 20.0x_4}{f_2^*} & \leq 0
\end{align*}
\] (68)

\[
\begin{align*}
x_1 + x_2 + x_3 + x_4 & \leq 15 \\
7.0x_1 + 5.0x_2 + 3.0x_3 & + 2.0x_4 & \leq 120 \\
3x_1 + 5x_2 + 10x_3 + 15x_4 & \leq 100 \\
x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0
\end{align*}
\] (69)

**Step 5. Solving the \( \lambda \)-problem.**

Results of solving the \( \lambda \)-problem, optimum values of variables:

\[
X^0 = X_0 = \{L_0 = 0.92179, x_1 = 0, x_2 = 11.739, x_3 = 1.5207, x_4 = 1.7396\} \] (70)

optimum value of criterion function \( \lambda^o = 0.92179 \).

We will execute a check at an optimum point of \( X^o = \{x_1, x_2, x_3, x_4\} \) and determine the sizes of criterion functions of \( f_k^o = \{f_{1k}(x^o), k = \overline{1,K}\} \) and relative estimates of \( \lambda_k^o = \{\lambda_{1k}(X^o), k = \overline{1,K}\} \). As a result, we will obtain:

\[
f(X_0) = \{f_1(X_0) = 91.52, f_2(X_0) = 161.3\}, \lambda_1(X^o) = 0.9218, \lambda_2(X^o) = 0.9218, \text{ i.e. } \lambda_1^o \leq \lambda_2^o(X^o), k = 1,2.
\]

These results showed that, at the \( X^o \) point, both criteria at the relative units reached the \( \lambda^o = 0.9217 \) level from the optimum sizes. Any increase in one of the criteria at this level leads to a decrease in another criterion, i.e., the point of \( X^o \) is Pareto-optimal.

In the considered VPLP with two homogeneous criteria - the model of the economic system, we obtained the optimum point \( X^o \) and maximum relative estimation \( \lambda^o : X^o = \{x_1 = 0, x_2 = 11.739, x_3 = 1.5207, x_4 = 1.7396\}, \lambda^o = 0.9218 \). At the optimum point \( X^o \), the criteria in natural units are: \( f_1(X^o) = 91.52, f_2(X^o) = 370.8 \), and this result confirmed the proofs of theorem 1, i.e., \( \lambda^o = \lambda_1(X^o) = \lambda_2(X^o) = 0.9218, 1, 2 \in K, X^o \in S \).

Theoretically, the point \( X^o \) is the center of symmetry. Indeed, the \( X^o \) point lies between the points of the optimum \( X_1^1 \) and \( X_2^2 \) obtained for each criterion (step 1), in which \( \lambda_1(X^o) = 1, \lambda_2(X^o) = 1 \). Let's show geometrically symmetry in **Figure 4**.

\[
\begin{align*}
\lambda_1(X_1^1) &= 1 \quad X_1^1 \\
\lambda_2(X_2^2) &= \lambda_2(X^o) = 0.9218
\end{align*}
\]

**Figure 4. Geometric interpretation of symmetry in modeling of an economic system with normalized criteria: \( \lambda_1(X), \lambda_2(X) \).**

Since there are two criteria, this example considers even numerical symmetry.

### 3.1.1.3 Present the Text of the Program in the Matlab Vector Problem in Linear Programming

% Vector linear in programming problem-two criteria.
% Author: Mashunin Yu. K. - Mashunin Yury. K.
% The program is designed for training and research, for commercial % purposes please contact: Mashunin@mil.ru

disp('Vector linear programming problem - 2 criteria')
disp(' max f(X) = \{max f_1(X) = (4x_1 + 5x_2 + 9x_3 + 11x_4), \}')
disp(' max f_2(X) = (2.0 x_1 + 10.0x_2 + 6.0 x_3 + 20.0 x_4, \}')
The conditions of a certainty in the VPMP are characterized by the fact that the functional dependence of each characteristic and restrictions on the parameters of the studied object is known\(^{[6-18]}\).

The conditions of uncertainty in the VPMP are characterized by the fact that the initial data characterizing the studied object are presented: a) random, b) fuzzy, or, c) incomplete data. Therefore, we lack sufficient information about the functional dependence of each characteristic and restrictions on the parameters in the VPMP\(^{[6-18]}\). For options a) and b) basic data have to be transformed to option c) and are presented in the table form. In work the option c)-with not full data which are, as a rule, obtained from experimental data is investigated.

In real life, the conditions of certainty and uncertainty are combined. The process model should also reflect these conditions. We will present a model of the engineering system in the VPMP under certainty and uncertainty in the aggregate:

\[
\text{Opt } F(X) = \{ \text{max } F_1 (X) = \{ \text{max } f_k (X), k = 1, K_1^{\text{def}} \}, \quad (71)
\]
\[
\text{max } I_1 (X) \equiv \{ \text{max } f_k (X_i, i = 1, M)_T, k = 1, K_1^{\text{unidef}} \}, \quad (72)
\]
\[
\text{min } F_2 (X) = \{ \text{min } f_k (X), k = 1, K_2^{\text{def}} \}, \quad (73)
\]
\[
\begin{align*}
\min l_2(X) & \equiv \{ \min f_k(X, i = 1, M) \}^T, \quad k = 1, K_1^{\text{mec}} \} \tag{74}
\end{align*}
\]

At restrictions,
\[
\begin{cases}
f_k^{\min} \leq f_k(X) \leq f_k^{\max} \\
\quad k = 1, K \\
\chi_j^{\min} \leq \chi_j \leq \chi_j^{\max} \\
\quad j = 1, N
\end{cases}
\]  \tag{75}

where \( X = \{x_j, j = 1, N \} \) is the vector input parameters of the studied object (of controlled variables); \( F(X) = \{ F_1(X) \} \) \( F_2(X) I_1(X) I_2(X) \) is a vector criterion, each component of which represents output characteristics of the studied object (a vector of criteria). The magnitude of the characteristic (function) depends on the discrete values of the vector of variables \( X \). \( F_1(X) F_2(X) \) is the set of functions max and min, respectively; \( I_1(X) I_2(X) \) are the set of discrete values of the characteristics max and min, respectively; \( 1, K_1^{\text{def}}, 1, K_2^{\text{def}} \) (definiteness) and \( 1, K_1^{\text{mec}}, 1, K_2^{\text{mec}} \) (uncertainty) a set of criteria max and min formed under certainty and uncertainty; in (61), \( f_k^{\min} \leq f_k(X) \leq f_k^{\max}, k = 1, K \) is a vector-function of restrictions imposed on the operation of the technological process, \( \chi_j^{\min} \leq \chi_j \leq \chi_j^{\max} \), \( j = 1, N \) are parametric restrictions imposed on the studied object.

3.1.2.3 Transformation of a Problem of Decision-making in the Conditions of Uncertainty into a VPMP in the Conditions of Certainty

Uncertainty transformation consists in the use of qualitative and quantitative descriptions of the object under study, for example, on the principle of "input-output". Quantitative data of the object under study (engineering system) can be obtained as a result of experimental research. In this case, the matrix of experimental studies will take the form:

\[
I = [X, Y] = \{ x_{i1}, x_{i2}, ..., x_{i(N-1)}, x_{iN}; y_{i1}, y_{i2}, ..., y_{i(K-1)}, y_{iK}; i = 1, M \} \tag{76}
\]

Where \( |X| = \{ x_{i1}, x_{i2}, ..., x_{i(N-1)}, x_{iN}; i = 1, M \} \) are parameters of the engineering system, \( N \)-many engineering system parameters; \( |Y| = \{ y_{i1}, y_{i2}, ..., y_{i(K-1)}, y_{iK}; i = 1, M \} \) these are the characteristics of the engineering system, \( K \)-many characteristics (criteria) of ES.

Information transformations \( I = [X, Y] \) that represent the initial data, in the functional dependence is carried out by using mathematical methods (regression analysis)\(^{[8-19]}\).

The transformation of the vector function (criteria) is carried out by the method of least squares, \( \min \sum_{i=1}^{M} (y_i - \bar{y}_i)^2 \), where by \( y_i, i = 1, M \) - experimental data; \( \bar{y}_i, i = 1, M \), their sizes received for one-factorial model by means of function \( \bar{y}_i = f(X_i, A) \). \( X_i = \{x_i \} \). The function \( f(X, A) \) we present in the form of a polynomial. In the applied part of the polynomial of the second degree is used.

As a result of this transformation, the source data \( I = [X, Y]; \{ f_k(X_i, i = 1, M) \}^T, k = 1, K_1^{\text{mec}}, \{ f_k(X_i, i = 1, M) \}^T, k = 1, K_2^{\text{mec}} \) in problems of decision-making in the conditions of uncertainty the functions: \( f_k(X), k = 1, K_1^{\text{mec}}, f_k(X), k = 1, K_2^{\text{mec}} \) are transformed.

As a result, a vector problem with conditions of certainty and uncertainty Equations (57)-(60) is transformed into a vector problem of mathematical programming under conditions of certainty:

\[
\begin{align*}
\text{Opt } F(X) & = \{ \max F_1(X) = \{ \max f_k(X) \} k = 1, K_1 \}, \tag{76} \\
\min F_2(X) & = \{ \min f_k(X) \} k = 1, K_2 \} \tag{77}
\end{align*}
\]

at restrictions,
\[
\begin{cases}
f_k^{\min} \leq f_k(X) \leq f_k^{\max} \\
\quad k = 1, K \\
\chi_j^{\min} \leq \chi_j \leq \chi_j^{\max} \\
\quad j = 1, N
\end{cases}
\]  \tag{75}
where \( F(X) = \{ f_k, k = 1, K \} \) is a vector criterion, each component of which represents a characteristic of the object under study, functionally dependent on the vector of variables \( X; F(X) = \{ F_1(X), F_1(X) \}; \) a subset of the criteria in the conditions of certainty:

\[
K_1 = K_1^{\text{def}} U K_1^{\text{unc}} \quad (78)
\]

in the conditions of uncertainty:

\[
K_2 = K_2^{\text{def}} U K_2^{\text{unc}} \quad (79)
\]

The VPMP (64)-(65) is analogous to VPMP (1)-(4)\([19]\).

3.1.2.4 Methodology for Selecting Optimal Parameters of Engineering Systems in Conditions of Certainty and Uncertainty Based on Vector Optimization

As an object of research, we consider: Engineering systems, which include "technical systems", "technological processes", "materials"\([17,18]\). The study of the engineering system is presented both with conditions of certainty and uncertainty. Mathematical software is based on the methods of vector optimization presented in the third section. Methodological support for modeling the engineering system is formed as: "Methodology for selecting the optimal parameters of Engineering systems in conditions of certainty and uncertainty." The methodology consists of six stages.

1. Formation of initial data (technical specification) for numerical modeling, selection of optimal parameters of the engineering system. The technical specification is formed by a designer or researcher who designs an engineering system.
2. Construction of mathematical and numerical models of the engineering system in conditions of certainty and uncertainty.
3. Solution of the Vector Problem of Mathematical Programming (VPMP), which represents the model of the engineering system with equivalent criteria.
4. Geometric interpretation of the results of the solution VPMP in a three-dimensional coordinate system in relative units.
5. Solution of the VPMP - the model of the engineering system, for which the priority of one of the criteria is set.
6. Geometric interpretation of the results of the solution in two- and three-dimensional coordinate systems in physical units.

3.1.3 Modeling and Selection of Optimal Parameters of a Technical System in Conditions of Certainty and Uncertainty Based on VPMP: Design Automation

The problem of numerical modeling and simulation of a technical system in which data, first, on a certain set of functional characteristics - conditions of certainty, second, discrete values of characteristics - conditions of uncertainty and restrictions imposed on the functioning of the technical system are known is considered\([6,8,16]\). The numerical problem of modeling a technical system is considered with equivalent criteria and with a given criterion priority.

3.1.3.1 Stage 1: Formation of Initial Data (Technical Specifications)

It is given. We're investigating the engineering (technical) system. The functioning of the technical system is determined by four parameters \( X=\{x_1, x_2, x_3, x_4\} \), which represent the vector of controlled variables. The parameters of the technical system are set within the following limits:

\[
\begin{align*}
22.0 & \leq x_1 \leq 88.0 \\
0 & \leq x_2 \leq 66.0 \\
2.2 & \leq x_3 \leq 8.8 \\
2.20 & \leq x_4 \leq 8.80 
\end{align*}
\quad (80)
\]

The operation of the technical system is determined by four criteria (characteristics): \( F(X)=\{f_1(X), f_2(X), f_3(X), f_4(X)\} \); the size of an assessment depends on a vector of \( X \).

The conditions of certainty in the technical system. For four characteristics of \( f_4(X) \) functional dependence on parameters \( X = \{x_j, j = 1, N\} \) is known, \( N=4 \).

\[
f_4(X) = 19.25 - 0.0081 * x_1 - 0.7005 * x_2 - 0.3605 * x_3 + 0.9769 * x_4 + 0.0126 * x_1 * x_2 + 0.0644 * x_1 * x_3 - 0 * x_1 * x_4 + 0.0396 * x_2 * x_3 + 0.0002 * x_2 * x_4 + 0.0004 * x_3 * x_4 - 0.0016 * x_1^2 + 0.0027 * x_2^2 + 0.0045 * x_3^2 - 0.0235 * x_4^2 \quad (81)
\]
The uncertainty condition in the technical system. For the first $f_1(X)$, second $f_2(X)$ and third $f_3(X)$ characteristic the results of experimental data are known: the values of the parameters and corresponding characteristics. Numerical values of parameters $X$ and characteristics of $y_1(X)$, $y_2(X)$ and $y_3(X)$ are presented in Table 2.

Table 2. Numerical Values of Parameters $X$ and Characteristics $y_1(X)$, $y_2(X)$ and $y_3(X)$ of the System

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<th>$x_4$</th>
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<td>88</td>
<td>66</td>
<td>2.2</td>
<td>5.5</td>
<td>547.8</td>
<td>1144.0</td>
<td>56.1</td>
</tr>
<tr>
<td>88</td>
<td>66</td>
<td>2.2</td>
<td>8.8</td>
<td>553.3</td>
<td>1166.0</td>
<td>55.0</td>
</tr>
<tr>
<td>88</td>
<td>66</td>
<td>5.5</td>
<td>2.2</td>
<td>561.0</td>
<td>1208.9</td>
<td>53.0</td>
</tr>
<tr>
<td>88</td>
<td>66</td>
<td>5.5</td>
<td>5.5</td>
<td>569.8</td>
<td>1232.0</td>
<td>50.6</td>
</tr>
<tr>
<td>88</td>
<td>66</td>
<td>5.5</td>
<td>8.8</td>
<td>575.3</td>
<td>1276.0</td>
<td>48.4</td>
</tr>
<tr>
<td>88</td>
<td>66</td>
<td>8.8</td>
<td>2.2</td>
<td>583.0</td>
<td>1303.5</td>
<td>47.7</td>
</tr>
<tr>
<td>88</td>
<td>66</td>
<td>8.8</td>
<td>5.5</td>
<td>595.1</td>
<td>1342.0</td>
<td>44.0</td>
</tr>
</tbody>
</table>
Tsessment size of the first and the third characteristic (criterion) could be possible received above: $f_1(X)\rightarrow\max$ $f_3(X)\rightarrow\max$; for the second and fourth characteristic is possible below: $y_2(X)\rightarrow\min$ $f_4(X)\rightarrow\min$. Parameters $X=\{x_1, x_2, x_3, x_4\}$ change in the following limits:

$$
\begin{cases}
x_1\in[22.55,88] \\
x_2\in[0.33,66] \\
x_3\in[2.20,50,88] \\
x_4\in[2.20,50,88]
\end{cases} \quad (82)
$$

To construct a model of the system in the form of a vector problem. To solve a vector problem with equivalent criteria. To choose priority criterion. To establish numerical value of priority criterion. To make the best decision with a specified priority criterion. Matlab system the software was developed for the decision of the vector problem of mathematical programming. The vector problem includes four variables (parameters of technical system): $X=\{x_1, x_2, x_3, x_4\}$ and four criteria (characteristic) of $F(X)=\{f_1(X), f_2(X), f_3(X), f_4(X)\}$. But for each new data (new system) the program is configured individually. In the software criteria of $F(X)=\{f_1(X), f_2(X), ..., f_6(X)\}$ with uncertainty conditions (in Table 2 they are provided as a part of $\{y_1,y_2,y_3,y_4\}$ can change from zero (i.e., all criteria are constructed in the conditions of determinacy) to six (i.e., all criteria are constructed in the conditions of uncertainty).

### 3.1.3.2 Stage 2: Creation of Mathematical and Numerical Model of the System in the Conditions of Definiteness and Indeterminacy

Creating a numerical model of the technical system includes the following sections: choosing a mathematical model of the system; Building a model in certainty conditions; construction in the conditions of not certainty; construction of a numerical model of the technical system under certainty and uncertainty.

Mathematical model of the technical system we will present model of the system in the conditions of definiteness and uncertainty in total in the form VPMP, as Equations (71)-(75):

$$
\text{Opt } F(X) = \{ \max F_1(X) = \{ \max f_k(X), k = 1, K_1^{\text{def}} \} \} \quad (71)
$$

$$
\text{max } I_1(X) \equiv \{ \max f_k(X, i = 1, M) \}^T, k = 1, K_1^{\text{unc}} \} \quad (72)
$$

$$
\text{min } F_2(X) = \{ \min f_k(X), k = 1, K_2^{\text{def}} \} \quad (73)
$$

$$
\text{min } I_2(X) \equiv \{ \min f_k(X, i = 1, M) \}^T, k = 1, K_1^{\text{unc}} \} \quad (74)
$$

At restrictions,

$$
\begin{cases}
f_k^{\text{min}} \leq f_k(X) \leq f_k^{\text{max}} \\
k = 1, K
\end{cases}
$$

$$
\begin{cases}
x_j^{\text{min}} \leq x_j \leq x_j^{\text{max}} \\
j = 1, N
\end{cases} \quad (75)
$$

$x_j^{\text{min}}, x_j^{\text{max}}, j = 1, N$ is vector of operated variable (design data); $F(X)=\{F_1(X), F_2(X), I_1(X), I_2(X)\}$ is vector criterion (71)-(74) which everyone a component represents a vector of characteristics (criteria) of the system which functionally depend on discrete values of a vector of variables $X = \{x_j, j = 1, N\};$

$F_1(X) = \{f_k(X), k = 1, K_1^{\text{def}} \}, \quad F_2(X) = \{f_k(X), k = 1, K_2^{\text{def}} \}$ are a set of the max and min functions respectively;

$I_1(X) = \{(f_k(X, i = 1, M))^T, k = 1, K_1^{\text{unc}} \}, \quad I_2(X) = \{(f_k(X, i = 1, M))^T, k = 1, K_2^{\text{unc}} \}$ are set of matrices of max and min respectively; $K_1^{\text{def}}, K_2^{\text{def}}$ (definiteness), $K_1^{\text{unc}}, K_2^{\text{unc}}$ (uncertainty) the set of criteria of $\max$ and $\min$ created
in the conditions of definiteness and uncertainty; in (75) \( f_k^{\text{min}} \leq f_k(X) \leq f_k^{\text{max}}, k = 1, K \) is a vector function of the restrictions imposed on functioning of technical system; \( x_j^{\text{min}} \leq x_j \leq x_j^{\text{max}}, j = 1, N \) - parametric restrictions.

It was assumed that the functions \( f_k(X), k = 1, K \) are differentiable and convex, \( g_i(X), i = 1, M \) are continuous, and the set of admissible points \( S \) given by constraints (8) is non-empty and is a compact: \( S = \{X \in \mathbb{R}^n | G(X) \leq 0, X^{\text{min}} \leq X \leq X^{\text{max}} \} \neq \emptyset \).

Construction in conditions of certainty is determined by the functional dependence of each characteristic and constraints on the parameters of the technical system. In our example, characteristic (80) and constraints (81) are known. Using data we construct a two-criterion vector problem of nonlinear programming in conditions of certainty:

\[
\min f_4(X) = 19.25 - 0.0081 * x_1 - 0.7005 * x_2 - 0.3605 * x_3 + 0.9769 * x_4 + 0.0126 * x_1 * x_2 + 0.6644 * x_1 * x_3 - 0 * x_1 * x_4 + 0.0396 * x_2 * x_3 + 0.0002 * x_2 * x_4 + 0.0004 * x_3 * x_4 - 0.0016 * x_1^2 + 0.0027 * x_2^2 + 0.0045 * x_3^2 - 0.0235 * x_4^2 \quad (81)
\]

\[
\begin{align*}
22.0 & \leq x_1 \leq 88.0 \\
0 & \leq x_2 \leq 66.0 \\
2.2 & \leq x_3 \leq 8.8 \\
2.20 & \leq x_4 \leq 8.80
\end{align*}
\]

These data (80)-(81) are used further at creation of a mathematical model of technical systems. Construction in the conditions of not certainty.

Construction in the conditions of uncertainty consists in the use of the qualitative and quantitative descriptions of technical systems received by the principle "input-output" in Table 2. Transformation of information (basic data of \( y_i(X), y_2(X) \) and \( y_3(X) \)) to a functional type of \( f_i(X), f_2(X), f_3(X) \) was carried out by use of mathematical methods (the regression analysis). Basic data of Table 1 are created in MATLAB system in the form of a matrix:

\[
I = |X, Y| = \{x_{11}, x_{12}, ..., x_{i(N-1)}, x_{iN}; y_{11}, y_{12}, ..., y_{i(k-1)}, y_{iK}, i = 1, M\} \quad (76)
\]

For each set, experiment with these \( y_2, y_3 \) function of regression on a method of the smallest squares minimization \( \min \sum_{i=1}^{M} (y_i - \hat{y}_i)^2 \) in MATLAB. \( A_k \) polynomial defining interrelation of the parameters \( X = \{x_{11}, x_{12}, x_{13}, x_{4}\} \) and functions \( \hat{y}_i = f(X, A_k), k = 2, 3 \) is for this purpose formed. As a result of calculations we receive system of coefficients of \( A_k = \{A_{0k}, A_{1k}, ..., A_{4k}\} \) which define coefficients of quadratic a polynomial (function):

\[
f_k(X, A) = A_{0k} + A_{1k}x_1 + A_{2k}x_2 + A_{3k}x_3 + A_{4k}x_4 + A_{5k}x_1 \cdot x_2 + A_{6k}x_1 \cdot x_3 + A_{7k}x_1 \cdot x_4 + A_{8k}x_2 \cdot x_3 + A_{9k}x_2 \cdot x_4 + A_{10k}x_3 \cdot x_4 + A_{11k}x_1^2 + A_{12k}x_2^2 + A_{13k}x_3^2 + A_{14k}x_4^2, k = 1, 2, 3 \quad (83)
\]

As a result of calculations of coefficients of \( A_k, k = 3 \), we received the \( f_1(X), f_2(X) \) and \( f_3(X) \) function:

\[
f_1(X) = 296.85 - 1.874 \cdot x_1 - 2.911 \cdot x_2 + 8.939 \cdot x_3 + 10.936 \cdot x_4 + 0.0734 \cdot x_1 \cdot x_2 - 0.0047 \cdot x_1 \cdot x_3 - 0.0128 \cdot x_1 \cdot x_4 + 0.0563 \cdot x_2 \cdot x_3 - 0.0789 \cdot x_2 \cdot x_4 - 0.0025 \cdot x_3 \cdot x_4 + 0.0108 \cdot x_1^2 - 0.0089 \cdot x_2^2 - 0.1844 \cdot x_3^2 - 0.3808 \cdot x_4^2 \quad (84)
\]

\[
f_2(X) = 875.3 + 23.893 \cdot x_1 - 30.866 \cdot x_2 - 25.858 \cdot x_3 - 45 \cdot x_4 - 0.6984 \cdot x_1 \cdot x_2 + 0.4276 \cdot x_1 \cdot x_3 + 0.6793 \cdot x_1 \cdot x_4 - 0.1167 \cdot x_2 \cdot x_3 + 0.2969 \cdot x_2 \cdot x_4 + 0.0093 \cdot x_3 \cdot x_4 + 0.0362 \cdot x_1^2 + 0.0331 \cdot x_2^2 + 2.9158 \cdot x_3^2 + 2.4052 \cdot x_4^2 \quad (85)
\]

\[
f_3(X) = 43.734 + 0.6598 \cdot x_1 + 0.4493 \cdot x_2 - 0.3094 \cdot x_3 - 1.8334 \cdot x_4 - 0.01 \cdot x_1 \cdot x_2 - 0.0062 \cdot x_1 \cdot x_3 + 0.0146 \cdot x_1 \cdot x_4 - 0.013 \cdot x_2 \cdot x_3 + 0.0121 \cdot x_2 \cdot x_4 - 0.0004 \cdot x_3 \cdot x_4 - 0.0003 \cdot x_1^2 - 0.0002 \cdot x_2^2 + 0.00254 \cdot x_3^2 + 0.0939 \cdot x_4^2 \quad (86)
\]

The minimum and maximum values of experimental data \( y_1(X), y_2(X) \) and \( y_3(X) \) are presented in the lower part of Table 2. The minimum and maximum values of the functions \( f_1(X), f_2(X), f_3(X) \) slightly differ from experimental data. The index of correlation and coefficients of determination are presented in the lower lines of Table 2. Results of the regression analysis \((83)-(84)\) are used further at creation of mathematical model of technical system.

Construction of a numerical model of the system under certainty and uncertainty.
For the creation of a numerical model of the system we used: the functions received conditions of definiteness (81) and uncertainty (85)-(86); parametric restrictions (80). A set of criteria $K=4$ included two criteria of $f_i(X)\rightarrow \text{max}$, $f_i(X)\rightarrow \text{min}$ and two criteria of $f_j(X)\rightarrow \text{min}$, $f_j(X)\rightarrow \text{min}$.

As a result, model of functioning of the system was presented a vector problem of mathematical programming:

$$\text{opt } F(X) = \{ \text{max } F_1(X) = \{ \text{max } f_1(X)\equiv 296.8 - 1.874 \times x_1 - 2.911 \times x_2 + 8.939 \times x_3 + 10.936 \times x_4 + 0.0734 \times x_1 \times x_2 - 0.0047 \times x_1 \times x_3 - 0.0128 \times x_1 \times x_4 + 0.0563 \times x_2 \times x_3 - 0.0789 \times x_2 \times x_4 - 0.0025 \times x_3 \times x_4 + 0.0108 \times x_1^2 + 0.0089 \times x_2^2 - 0.1844 \times x_3^2 - 0.3808 \times x_4^2 \} \}$$

$$\text{opt } F_2(X) = \{ \text{max } f_2(X)\equiv 43.734 + 0.6598 \times x_1 + 0.4493 \times x_2 - 0.3094 \times x_3 - 1.8334 \times x_4 - 0.01 \times x_1 \times x_2 - 0.0062 \times x_1 \times x_3 + 0.0146 \times x_1 \times x_4 - 0.013 \times x_2 \times x_3 + 0.0121 \times x_2 \times x_4 - 0.0004 \times x_3 \times x_4 - 0.0003 \times x_1^2 - 0.0002 \times x_2^2 + 0.00254 \times x_3^2 + 0.0939 \times x_4^2 \} \}$$

$$\text{min } F_1(X) = \{ \text{min } f_1(X)\equiv 875.3 + 23.893 \times x_1 - 30.866 \times x_2 - 25.858 \times x_3 - 45 \times x_4 - 0.6984 \times x_1 \times x_2 + 0.4276 \times x_1 \times x_3 + 0.6793 \times x_1 \times x_4 - 0.1167 \times x_2 \times x_3 + 0.2969 \times x_2 \times x_4 - 0.0093 \times x_3 \times x_4 + 0.0362 \times x_1^2 + 0.0331 \times x_2^2 + 2.9158 \times x_3^2 + 2.4052 \times x_4^2 \}$$

$$\text{min } F_4(X) = \{ \text{min } f_4(X)\equiv 19.253 - 0.0081 \times x_1 - 0.7005 \times x_2 - 0.3605 \times x_3 + 0.9769 \times x_4 + 0.0126 \times x_1 \times x_2 + 0.0644 \times x_1 \times x_3 - 0 \times x_1 \times x_4 + 0.0396 \times x_2 \times x_3 + 0.0002 \times x_2 \times x_4 + 0.0004 \times x_3 \times x_4 - 0.0016 \times x_1^2 + 0.0027 \times x_2^2 + 0.0045 \times x_3^2 - 0.0235 \times x_4^2 \}$$

At restrictions:

$$22.0 \leq x_1 \leq 88.0$$

$$0 \leq x_2 \leq 66.0$$

$$2.2 \leq x_3 \leq 8.8$$

$$2.20 \leq x_4 \leq 8.80$$

(80)

The vector problem of mathematical programming (87)-(90) represents the model decision making under certainty and uncertainty in the aggregate.

3.1.3.3 Stage 3: The Solution of the VPMP-Model of the System at Equivalent Criteria

To solve the VPMP (76)-(80), methods based on the axioms of the normalization of criteria and the principle of guaranteed results are presented, which follow from axiom 1 and the principle of optimality 1.

The solution of a vector problem of mathematical programming (76)-(80) which was submitted as a sequence of steps.

Step 1. VPMP (76)-(80) were solved by each criterion separately, thus using the function fmincon (…) of Matlab system, the appeal to the function fmincon (…) is considered in these studies. As a result of calculation for each criterion we received optimum points: $X^*_k = f_k(X^*_k), k = 1, K$-sizes of criteria in this point, i.e., the best decision on each criterion in the VPMP:

$$X_1^k = \{ x_1 = 88, x_2 = 66, x_3 = 8.8, x_4 = 2.20 \}$$

$$X_2^k = \{ x_1 = 22, x_2 = 0.0, x_3 = 2.83, x_4 = 6.25 \}$$

$$X_3^k = \{ x_1 = 22, x_2 = 0.0, x_3 = 2.20, x_4 = 8.80 \}$$

$$X_4^k = \{ x_1 = 22, x_2 = 62.17, x_3 = 2.20, x_4 = 2.20 \}$$

$$f_1^k = f_1(X_1^k) = - 535.06$$

$$f_2^k = f_2(X_2^k) = 1301.2$$

$$f_3^k = f_3(X_3^k) = - 100.15$$

$$f_4^k = f_4(X_4^k) = 12.247$$

(91)

With constraints (80), we will represent in a two-dimensional coordinate system $\{x_1, x_2\}$ on the Figure 5A: First, the allowable set of points $S$, formed by constraints $22.0 \leq x_1 \leq 88.0, 0 \leq x_2 \leq 66.0$; Second, the Pareto set $S^\prime$, which includes the path formed by the points $X_1^1, X_2^2, X_3^3, X_4^4$, $S^\prime \subset S$; third, optimum points $X_1^k, X_2^k, X_3^k, X_4^k$; fourth, the optimum point $X^o$, obtained in the fourth step.
Step 2. We defined the worst unchangeable part of each criterion (anti-optimum):

\[
\begin{align*}
X_1^0 &= \{ x_1 = 22, x_2 = 66.0, x_3 = 2.2, x_4 = 2.2 \} \\
X_2^0 &= \{ x_1 = 88, x_2 = 0.0, x_3 = 8.8, x_4 = 8.8 \} \\
X_3^0 &= \{ x_1 = 22, x_2 = 0.0, x_3 = 8.8, x_4 = 8.07 \} \\
X_4^0 &= \{ x_1 = 88, x_2 = 66, x_3 = 8.80, x_4 = 8.80 \}
\end{align*}
\]

(93)

\[
\begin{align*}
f_1^0 &= f_1(X_1^0) = 243.25 \\
f_2^0 &= f_2(X_2^0) = -3903.1 \\
f_3^0 &= f_3(X_3^0) = 50.03 \\
f_4^0 &= f_4(X_4^0) = -121.83
\end{align*}
\]

(94)

Figure 5. The schematic diagrams of set of admissible points and solution of the \(\lambda\)-problem on the Basis of the Vector Optimization Problem. A: The set of admissible points \(S\) and Pareto optimal points \(S^*\in S, X_1^*, X_2^*, X_3^*, X_4^*\) in a two-dimensional system of coordinates \(\{x_1, x_2\}\); B: The solution of the \(\lambda\)-problem in a three-dimensional system of coordinates of \(x_1, x_2\) and \(\lambda\); C: The solution of the \(\lambda\)-problem (1, 3 criterion) in a three-dimensional system of coordinates of \(x_1, x_2\) and \(\lambda\). D: Geometric interpretation of symmetry in modeling of technical systems with normalized criteria: \(\lambda_1(X), \ldots, \lambda_4(X)\).
Step 3. Performed system analysis of a set of points, optimum across Pareto, i.e., the analysis for each criterion in points of an optimum of \( X^*\)={\( X_1^*, X_2^*, X_3^*, X_4^* \)} sizes of criterion functions of \( F(X^*) = \| f_q(X^*_k) \|_{q=1K}^{k=1K} \) are determined. Calculated a vector of \( F(d_1, d_2, d_3, d_4) \)- deviations by each criterion on an admissible set of \( S: d_k = f_k^* - f_k^0, k = 1,4 \), and matrix of relative estimates of

\[
\lambda(X^*) = \| \Lambda_q(X^*_k) \|_{q=1K}^{k=1K} (30)
\]

where

\[
\lambda_k(X) = (f_k^* - f_k^0)/d_k (95)
\]

\[
F(X^*) = \begin{bmatrix} 535.1 & 1731.9 & 58.1 & 117.0 \\ 317.6 & 1301.2 & 51.3 & 26.5 \\ 192.5 & 3614.3 & 100.2 & 24.6 \\ 244.0 & 2458.2 & 67.7 & 12.2 \end{bmatrix} (96)
\]

\[
d_k = \begin{bmatrix} 291.8 \\ -2602.0 \\ 50.12 \\ -109.58 \end{bmatrix} (97)
\]

\[
\lambda(X^*) = \begin{bmatrix} 1.0000 & 0.8345 & 0.1603 & 0.0443 \\ 0.2548 & 1.0000 & 0.0244 & 0.8697 \\ -0.1740 & 0.1110 & 1.0000 & 0.8870 \\ 0.0027 & 0.5553 & 0.3532 & 1.0000 \end{bmatrix} (98)
\]

The analysis of sizes of criteria in relative estimates in the VPMP (85)-(88) showed that in points of an optimum of \( X^*=\{X_1^*, X_2^*, X_3^*, X_4^* \} \) the relative assessment is equal to the unit. For other criteria, there is much less than the unit. It is required to find such parameters (points) at which relative estimates are closest to unit. The steps 4 and 5 are directed on the solution of this problem.

Step 4. Creation of a \( \lambda \)-problem is carried out in two stages. Originally the maximum problem of optimization with the normalized criteria is under construction:

\[
\lambda^o = \max_{\lambda \in S} \min_{k \in K} \lambda_k(X), G(X) \leq 0, X \geq 0 (13)
\]

which at the second stage was transformed to a standard problem of mathematical programming (\( \lambda \)-problem):

\[
\lambda^o = \max \lambda (19)
\]

at restrictions

\[
\lambda - \frac{f_1(X) - f_1^0}{f_1 - f_1^0} \leq 0
\]

\[
\lambda - \frac{f_2(X) - f_2^0}{f_2 - f_2^0} \leq 0 (39)
\]

\[
\lambda - \frac{f_3(X) - f_3^0}{f_3 - f_3^0} \leq 0
\]

\[
\lambda - \frac{f_4(X) - f_4^0}{f_4 - f_4^0} \leq 0
\]

\[
0 \leq \lambda \leq 1
\]

\[
22 \leq x_1 \leq 88
\]

\[
0 \leq x_2 \leq 66 \quad (80)
\]

\[
2.2 \leq x_3 \leq 8.8
\]

\[
2.2 \leq x_4 \leq 8.8
\]

where the vector of unknown had dimension of \( N+1: X = \{x_1, x_2, x_3, x_4, \lambda \} \); the functions \( f_i(X), f_2(X), f_3(X) \) and \( f_4(X) \) correspond \( \text{Equations (84)-(85)} \) respectively. Substituting the numerical values of the functions \( f_1(X), f_2(X), f_3(X) \) and \( f_4(X) \), we get the \( \lambda \)-problem of the following form:

at restrictions
\[ \begin{align*}
\lambda & = \frac{-296.8-1.874\times x_1-2.91\times x_2-0.184\times x_3^2-0.38\times x_4^2-x_4^4}{f_1-f_3^4} \leq 0 \\
\lambda & = \frac{43.734+0.6598\times x_1+0.449\times x_2-0.000254\times x_3^2+0.0939\times x_4^2-x_4^4}{f_3^2-f_4^4} \leq 0 \\
\lambda & = \frac{-875.3+23.9\times x_1-30.8\times x_2+2.9158\times x_3^2+2.4052\times x_4^2-x_4^4}{f_2^2-f_1^4} \leq 0 \\
\lambda & = \frac{-19.253-0.0001\times x_1-7.005\times x_2-0.0045\times x_3^2-0.0235\times x_4^2-x_4^4}{f_1^2-f_4^4} \leq 0
\end{align*} \] (99)

Appeal to function fmincon(...):

\[ [X_0, Lo] = \text{fmincon}(\text{'Z_TehnSist_AKrit_L'}, X0, A0, b0, Aeq, beq, lbo, ubo, 'Z_TehnSist_LConst', options) \]

As a result of the solution of a vector problem of mathematical programming Equations (88)-(92) at equivalent criteria and \( \lambda \)-problem corresponding to it (87)-(90) received:

\[ X^o = \{X^o, \lambda^o\} = \{X^o = \{x_1 = 52.90, x_2 = 36.09, x_3 = 8.8, x_4 = 2.20, \lambda^o = 0.3179\} \] (100)

The optimum point – design data of the system, point \( X^o \) is presented in Figure 5A; \( f_k(X^o), k = 1, 2, 3 \)- sizes of characteristics (criteria) of technical system;

\[ \{f_1(X^o) = 336, f_2(X^o) = 2239, f_3(X^o) = 65.96, f_4(X^o) = 58.43\} \] (101)

\[ \lambda^o = 0.3179 \]

A relative assessment \( \lambda^o \) call the guaranteed result in relative units, i.e., \( \lambda_k(X^o) \) and according to the characteristic of the technical \( f_\lambda(X^o) \) system it is impossible to improve, without worsening thus other characteristics.

We will notice that according to the theorem 1, in \( X^o \) point criteria \( k=1, k=3 \) is contradictory. This contradiction is defined by equality of \( \lambda_k(X^o) = \lambda_k(X^o) = \lambda^o = 0.31791 \), and other criteria an inequality of \( \{\lambda_k(X^o) = 0.63941, \lambda_4(X^o) = 0.57851\} > \lambda^o \).

Thus, the theorem 1 forms a basis for determination of correctness of the solution of a vector problem of mathematical programming. In a VPMP (76)-(80), as a rule, for two criteria equality is carried out: \( \lambda^o = \lambda_4(X^o) = \lambda_4(X) \), \( q, p \in K, X \in S \), and for other criteria is defined as an inequality: \( \lambda^o \in S, \lambda_4(X^o), \forall k \in K, q \neq p \neq k \).

3.1.3.4 Stage 4: Creation of Geometrical Interpretation of Results of the Decision in the VPMP (76)-(80) in a Three-Dimensional Coordinate System in Relative Units

In an admissible set of points of \( S \) formed by restrictions (87), optimum points \( X^*_1, X^*_2, X^*_3, X^*_4 \) united in a contour, presented a set of points, optimum across Pareto, to \( S^c \subset S, \) Figure 5A.

Coordinates of these points, and also characteristics of the technical system in relative units of \( \lambda_1(X), \lambda_2(X), \lambda_3(X), \lambda_4(X) \) are shown in Figure 5B in three measured spaces, where the third axis of \( \lambda \) – a relative assessment.

Looking at a Figure 5B, we can provide changes of all functions of \( \lambda_1(X), \lambda_2(X), \lambda_3(X) \) and \( \lambda_4(X) \) in four measured spaces. We will consider, for example, an optimum point of \( X^*_3 \) The \( \lambda_1(X) \) function is created from the functions \( f_1(X) \) with variable coordinates \( \{x_1, x_2\} \) and with constant coordinates \( \{x_3 = 8.80, x_4 = 2.20\} \), taken from an optimum point \( X^o \). In a point \( X^1 \) the relative assessment of \( \lambda_1(X^1) = 0.831 \)-is shown in Figure 2B by a black point. But we know that the relative assessment of \( \lambda_1(X^1) \) received from the \( f_1(X^1) \) function on the third step is equal to unit, we will designate it as \( \lambda_1^1(X^1) = 1 \)-was shown in Figure 5B by a red point. The difference between \( \lambda_1^1(X^1) = 1 \) and \( \lambda_1(X) = 0.831 \) is an error \( \Delta = 0.171 \) transitions from four measured (and generally \( N \)-dimensional) to two-dimensional area.

The point \( X^*_1 \) and appropriate relative estimates of \( \lambda_1(X^*_1) \) and \( \lambda_4^1(X^*_1) \) is similarly shown.

Thus, for the first time in domestic and foreign practice transition and its geometrical illustration from \( N \)-dimensional to two-dimensional measurement of function is shown in vector problems of mathematical programming with the appropriate errors.
3.1.3.5 Stage 5: The Solution of a VPMP-model of the Technical System at the Given Priority of the Criterion

Step 1. Solve a vector problem of nonlinear programming with equivalent criteria.

The algorithm of the decision is presented in stage 3. Numerical results of the solution of the vector problem of nonlinear programming are given above. Pareto's great number of \( S^0 \subset S \) lies between optimum points \( X^*_1 \), \( X^*_2 \), \( X^*_3 \), \( X^*_4 \), \( X^*_5 \). We will carry out the analysis of a great number of Pareto \( S^0 \subset S \). For this purpose, we will connect auxiliary points: \( X_1, X_2, X_3, X_4, X_5 \) with a point \( X^0 \) which conditionally represented the center of a great number of Pareto. As a result, we have received four subsets of points \( X \in S^0 \subset S \), \( q = 4 \). The subset of \( S^0 \subset S \) is characterized by the fact that the relative assessment of \( \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \), i.e., in the field of \( S^0 \) first criterion has a priority over the others. Similar to \( S^0_1, S^0_2, S^0_3 \) - subsets of points where the second, third and fourth criterion has a priority over the others respectively. Set of points, optimum across Pareto we will designate \( S^0 = S^0_1 \cup S^0_2 \cup S^0_3 \cup S^0_4 \). Coordinates of all received points and relative estimates are presented in two-dimensional space \( \{x_1, x_2\} \) in Figure 5A. These coordinates are shown in three measured spaces \( \{x_1, x_2, \lambda\} \) in Figure 5B where the third axis of \( \lambda \) - a relative assessment. Restrictions of a set of points, optimum across Pareto, in Figure 5B it is lowered to -0.5 (that restrictions were visible). This information is also a basis for further research of the structure of a great number of Pareto. The person making decisions, as a rule, is the designer of the system. If results of the solution of a vector problem with equivalent criteria don't satisfy the person making the decision, then the choice of the optimal solution is carried out from any subset of points of \( S^0_1, S^0_2, S^0_3, S^0_4 \). These subsets of Pareto points are shown in Figure 5B in the form of functions \( f_i(X), f_2(X), f_3(X), f_4(X) \).

Step 2. Choice of priority criterion of \( q \in K \).

From the theory (see the theorem 1) it is known that in an optimum point of \( X^0 \) always there are two most inconsistent criteria, \( q \in K \) and \( p \in K \) for which in relative units exact equality is carried out: \( \lambda^0 = \lambda_4(X^0) = \lambda_p(X^0), q, p \in K, X \in S \), and for the others it is carried out inequalities: \( \lambda^0 \leq \lambda_k(X^0), \forall k \in K, q \neq p \neq k \).

In the model of the system and the corresponding \( \lambda \)-problem, such criteria are the first and third: \( \lambda^0 = \lambda_3(X^0) = \lambda_3(X^0) = 0.3179 \).

We will show the \( \lambda_3(X) \) and \( \lambda_3(X) \) functions separately in Figure 5C from an optimum point of \( X^0 = \{ X^0, \lambda^0 \} \).

Here all points and data about which it was told in Figure 5B were shown.

As a rule, the criterion which the decision-maker would like to improve gets out of couple of contradictory criteria. Such a criterion is called "priority criterion", we will designate it \( q = 3 \in K \). This criterion is investigated in interaction with the first criterion of \( q = 1 \in K \). We will allocate these two criteria from all sets of the criteria \( K = 4 \) shown in Figure 5D.

Step 3. Numerical limits of change of size of a priority criterion of \( q = 3 \in K \) are defined.

For priority criterion of \( q = 3 \) numerical limits in physical units upon transition from a point of an optimum of \( X^0 \) (6.35) to the point of \( X^*_q \) received on the first step are defined.

Information about the criteria for \( q = 3 \) are given on the screen:

\[
f_q(X^0) = 65.96 \leq f_q(X) \leq 100.15 = f_q(X^*_q), q \in K
\]

In relative units the criterion of \( q = 2 \) changes in the following limits:

\[
\lambda_q(X^0) = 0.31791 \leq \lambda_q(X) \leq 1 = \lambda_q(X^*_q), q = 3 \in K
\]

Step 4. Choice of size of priority criterion. \( q \in K \). (Decision-making).

The message is displayed: "Enter the size of priority criterion \( f_q = a \) - we enter, for example, \( f_q = 80 \).

Step 5. Calculation of a relative assessment.

For the chosen size of priority criterion of \( f_q = 80 \) the relative assessment is calculated:
\[
\lambda_q = \frac{f_q - f_q^0}{f_q^* - f_q^0} = \frac{80 - 50.03}{10.15 - 50.03} = 0.5979
\]

which upon transition from \(X^o\) point to \(X_q^*\) according to \(\lambda_q=\lambda_3(X^o)\) lies in limits:

\[
0.31791 = \lambda_3(X^o) \leq \lambda_3 = 0.5979 \leq \lambda_3(X_q^*) = 1, \ q \in K
\]

**Step 6. Calculation of coefficient of linear approximation.**

Assuming linear nature of change of criterion of \(f_k(X)\) and according to a relative assessment of \(\lambda_q(X)\), using standard methods of linear approximation, we will calculate proportionality coefficient between \(\lambda_q(X^o)\), \(\lambda_q\), which we will call \(\rho\):

\[
\rho = \frac{\lambda_q - \lambda_q(X^o)}{\lambda_q(X_q^*) - \lambda_q(X^o)} = \frac{0.5979 - 0.3179}{1 - 0.3179} = 0.4106, \ q = 3 \in K
\]

**Step 7. Calculation of coordinates of priority criterion with the size \(f_q\).**

Assuming linear nature of change of a vector of \(X^o=\{x_1, x_2\}, \ q=3\) we will determine coordinates of a point of priority criterion with the size \(f_q\) with a relative assessment \(\lambda_q^*:\)

\[
x_q^q = \{x_1 = X^o(1) + \rho(X_q^*(1) - X^o(1)), x_2 = X^o(2) + \rho(X_q^*(2) - X^o(2))\} \quad (104)
\]

\[
X^o = \{X^o(1) = 80.0, X^o(2) = 69.11\}
\]

\[
X_q^* = \{X_q^*(1) = 80.0, X_q^*(2) = 0.0\}
\]

As a result of calculations, we have received point coordinates:

\[
X^o=\{x_1=67.31, x_2=21.27\}
\]

**Step 8. Calculation of the main indicators of a point of \(X^o\).**

For the received \(X^o\) point, we will calculate:

all criteria in physical units

\[
f_k(X^o) = \{f_k(X^o), k = 1, K\}
\]

\[
f(X^o) = \{f_1(X^o) = 313.45, f_2(X^o) = 2575.7, f_3(X^o) = 74.2, f_4(X^o) = 60.6\}
\]

all relative estimates of criteria

\[
\lambda_q^* = \{\lambda_q^*, k = 1, K\}
\]

\[
\lambda_k(X^o) = \frac{f_k(X^o) - f_k^0}{f_k^* - f_k^0}, k = 1, K.
\]

\[
\lambda_k(X^o) = \{\lambda_1(X^o) = 0.2405, \lambda_2(X^o) = 0.5102, \lambda_3(X^o) = 0.4825, \lambda_4(X^o) = 0.5586\}
\]

minimum relative assessment:

\[
\min LX_q = \min(\lambda_k(X^o)) = 0.2405
\]

Vector of priorities

\[
P^q(X) = \{p_k^q = \frac{\lambda_k(X^o)}{\lambda^q(X^o)}, k = 1, K\}
\]

\[
P^q = \{p_1^q = 2.0061, p_2^q = 0.9458, p_3^q = 1.0, p_4^q = 0.8637\}
\]

Relative assessment taking into account a criterion priority:

\[
\lambda_k(X^o) \cdot P^q = \{p_1^3 \cdot \lambda_1(X^o) = 0.4825, p_2^3 \cdot \lambda_2(X^o) = 0.4825, p_3^3 \cdot \lambda_3(X^o) = 0.4825, p_4^3 \cdot \lambda_4(X^o) = 0.4825\}
\]

The minimum relative assessment taking into account a criterion priority:
\[
\lambda^o = \min (p_1^1 \lambda_1 (X^o), p_2^1 \lambda_2 (X^o), p_3^1 \lambda_3 (X^o), p_4^1 \lambda_4 (X^o)) = 0.4825
\]

Any point from Pareto's set \(X^o_1 = \{\lambda^o, X^o\} \in \mathcal{S}^o\) can be similarly calculated.

**Step 9. Analysis of the results of the solution of the model of the technical system represented by the vector problem from the current of symmetry.**

In the considered vector problem of nonlinear programming Equations (75)-(77) with four heterogeneous criteria - the model of the technical system and the corresponding numerical variant (87)-(90) and built on its basis \(\lambda\)-problems (91) we obtained the optimum point \(X^o\) and the maximum relative estimate \(\lambda^o\):

\[
X^o = \{X^o, \lambda^o\} = \{X^o = \{x_1 = 52.91, x_2 = 36.09, x_3 = 8.80, x_4 = 2.20, \lambda^o = 0.31791\}
\]

At the optimum point \(X^o\) the criteria in natural units are:

\[
F(X^o) = \{f_1(X^o) = 336, f_2(X^o) = 2239, f_3(X^o) = 65.96, f_4(X^o) = 58.43\}
\]

and relative units:

\[
\lambda (X^o) = \{\lambda_1 (X^o) = 0.31791, \lambda_2 (X^o) = 0.6394, \lambda_3 (X^o) = 0.31791, \lambda_4 (X^o) = 0.5785\}
\]

This result confirms the proofs of theorem 1 about the most contradictory criteria in the vector optimization problem, i.e.,

\[
\lambda^o = \lambda_1 (X^o) = \lambda_3 (X^o) = 0.31791, \{1, 3\} \in \mathcal{K}, X^o \in \mathcal{S}
\]

The rest of the criteria in relative units lie within:

\[
\{\lambda_2 (X^o) = 0.63941, \lambda_4 (X^o) = 0.57851\} > \lambda^o
\]

Theoretically, the point \(X^o\) is the center of symmetry. Indeed, \(X^o\) point lies between the points of the optimum \(X^o_1\) and \(X^o_3\) obtained for each criterion (step 1), in which \(\lambda_1 (X^o_1) = 1, \lambda_3 (X^o_3) = 1\). The geometrically symmetry was shown in Figure 5D.

For the two criteria of the first \(\lambda_1 (X)\) and the third \(\lambda_3 (X)\), this example considers an even numerical symmetry. The calculated size of the criterion \(f_q (X^o_1), q \in \mathcal{K}\) is usually not equal to the set \(f_q\). The error of the choice of \(\Delta f_q = |f_q (X^o_1) - f_q| = |74.2 - 80| = 5.8\) is defined by an error of linear approximation, \(\Delta f_q \approx 7.25\%\).

If error \(\Delta f_q = |f_q (X^o_1) - f_q| = |74.2 - 80| = 5.8\), measured in physical units or as a percentage \(\Delta f_q \approx \frac{\Delta f_q}{f_q} \times 100 = 7.25\%\), is more than set \(\Delta f, \Delta f > \Delta f\), we pass to a step 2, if \(\Delta f \leq \Delta f\), calculations come to the end.

In the course of modeling, parametric restrictions (19) can be changed, i.e., some set of optimum decisions is received. Choose a final version which in our example included from this set of optimum decisions: parameters of technical system:

\[
X^o = \{X^o, \lambda^o\} = \{X^o = \{x_1 = 52.91, x_2 = 36.09, x_3 = 8.80, x_4 = 2.20, \lambda^o = 0.31791\}
\]

the parameters of the technical system at a given priority criterion \(q=2\):

\[
X^q = \{x_1 = 67.32, x_2 = 21.26\}
\]

3.1.3.6 Geometrical Interpretation of Results of the Decision in a Three-dimensional Coordinate System in Physical Units

In the two-dimensional coordinate system \(x_1, x_2\) on Figure 5A, three-dimensional coordinate system \(x_1, x_2\) and \(\lambda\) in Figure 5B, \(f_1(X), f_2(X), f_3(X), f_4(X)\) were presented in physical units for each technical system characteristic (criterion).
The first characteristic of technical system \( f_1(X) \) in \( x_1, x_2 \) coordinates was shown in Figure 6A. Similarly, the same characteristic in relative units of \( \lambda_1(X) \) was shown in Figure 6B. Indicators \( f^1_2(X^1), f^2_2(X^2) \) of the first of characteristics of the system (are highlighted in red color) define transition errors from four-dimensional \( X=\{x_1, x_2, x_3, x_4\} \) to two-dimensional \( X^0=\{x_1, x_2\} \) to system of coordinates. The second characteristic of technical system \( f_1(X) \) in \( x_1, x_2 \) coordinates is shown in Figure 6C. Similarly, the same characteristic in relative units of \( \lambda_2(X) \) is shown in Figure 6D. Indicators of the second \( f^1_2(X^2), f^2_2(X^2) \) of characteristics of the system (are highlighted in red color) define transition errors from four-dimensional \( X^0=\{x_1, x_2, x_3, x_4\} \) to two-dimensional \( X^0=\{x_1, x_2\} \) to system of coordinates. The third characteristic of technical system \( f_1(X) \) in \( x_1, x_2 \) coordinates is shown in Figure 6E. Similarly, the same characteristic in relative units of \( \lambda_3(X) \) is shown in Figure 6F. Indicators of the third \( f^1_2(X^3), f^2_2(X^3) \) of characteristics of the system (are highlighted in red color) define transition errors from four-dimensional \( X^0=\{x_1, x_2, x_3, x_4\} \) to two-dimensional \( X^0=\{x_1, x_2\} \) to system of coordinates. The fourth characteristic of technical system \( f_1(X) \) in \( x_1, x_2 \) coordinates is shown in Figure 6G. Similarly, the same characteristic in relative units of \( \lambda_4(X) \) is shown in Figure 6H. Indicators of the fourth \( f^1_2(X^4), f^2_2(X^4) \) of characteristics of the system (are highlighted in red color) define transition errors from four-dimensional \( X^0=\{x_1, x_2, x_3, x_4\} \) to two-dimensional \( X^0=\{x_1, x_2\} \) to system of coordinates.

Collectively:
- point - \( X^0 \); characteristics of \( F(X^0)=\{f_1(X^0), f_2(X^0), f_3(X^0), f_4(X^0)\} \);
- relative estimates of \( \lambda(X^0)=\{\lambda_1(X^0), \lambda_2(X^0), \lambda_3(X^0), \lambda_4(X^0)\} \);
- maximum \( \lambda^o \) relative level such that \( \lambda^o \leq \lambda_k(X^0) \forall k \in K \).

There is an optimum decision at equivalent criteria (characteristics), and procedure of receiving is adoption of the optimum decision at equivalent criteria (characteristics).

- point-\( X^0 \); characteristics of \( F(X^0)=\{f_1(X^0), f_2(X^0), f_3(X^0), f_4(X^0)\} \);
- relative estimates of \( \lambda(X^0)=\{\lambda_1(X^0), \lambda_2(X^0), \lambda_3(X^0), \lambda_4(X^0)\} \);
- maximum \( \lambda^o \) relative level such that \( \lambda^o \leq \lambda_k(X^0), k = 1, K \)

There is an optimal solution at the set priority of the q-th criterion (characteristic) in relation to other criteria. Procedure of receiving a point is \( X^q \) adoption of the optimal solution at the set priority of the second criterion.

Theory of vector optimization, methods of solution of the vector problems with equivalent criteria and given priority of criterion can choose any point from the set of points, optimum across Pareto, and show the optimality of this point.
3.1.4 Modeling and Selection of Optimal Parameters of a Technological Process Under Conditions of Certainty and Uncertainty Based on Theory of Vector Optimization

We study a technological process for which data are known about a certain set of functional characteristics - certainty conditions, discrete values of characteristics - uncertainty condition and restrictions imposed on the functioning of the technological process. The numerical problem in the VPMP of modeling a technological process is solved with equivalent criteria and with a given priority of the criterion.

3.1.4.1 The Technical Assignment: The Choice of the Optimal Parameters of the Technological Process

The technological process, the operation of which is determined by two parameters: $X=\{x_1, x_2\}$ - the vector of variables (controlled), the operation of the process is determined by four characteristics (criteria): $F(X)=\{f_1(X), f_2(X), f_3(X), f_4(X)\}$, the value of which depends on the vector of parameters $X$.

Conditions of certainty. The conditions were characterized by the fact that the functional dependence of the fourth characteristic $f_4(X)$ on the parameters of the technological process $X=\{x_1, x_2\}$ was known:

$$f_4(X) = -0.2450 - 0.7470x_1 + 0.3832x_1^2 + 0.0442x_2 + 0.0012x_2^2 - 0.0346x_1x_2 \ (105)$$

Conditions of uncertainty. For the first, second and third characteristics of the technological process, the results of experimental data are known: the values of the parameters and the corresponding characteristics. The numerical values of the parameters $X$ and the characteristics $y_1(X), y_2(X), y_3(X)$ are presented in Table 3.

Table 3. Experimental Weld Input and Output Parameters

<table>
<thead>
<tr>
<th>Laser Power, $p$ (Analog V)</th>
<th>Travel Speed, $v$ (mm/sec)</th>
<th>Wire Feed Rate, $r$ (m/min)</th>
<th>Depth, $D$(mm)</th>
<th>Total Accumulated Pore Length, $P_o$ (mm/mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>$x_2$</td>
<td>$y_1(X)\rightarrow$ &amp; $\max$</td>
<td>$y_2(X)\rightarrow$ &amp; $\min$</td>
<td>$y_3(X)\rightarrow$ &amp; $\max$</td>
</tr>
<tr>
<td>2.40</td>
<td>24.2</td>
<td>4.2197</td>
<td>55.3951</td>
<td>-0.0365</td>
</tr>
<tr>
<td>2.76</td>
<td>18.72</td>
<td>3.2714</td>
<td>31.2497</td>
<td>0.0286</td>
</tr>
<tr>
<td>2.76</td>
<td>19.08</td>
<td>3.2770</td>
<td>32.3886</td>
<td>0.0271</td>
</tr>
<tr>
<td>2.76</td>
<td>31.68</td>
<td>4.2613</td>
<td>86.8526</td>
<td>0.0760</td>
</tr>
</tbody>
</table>
In the decision taken, the evaluation value for the first, second and third characteristics (criteria) are desirable to get as high as possible: \( y_1(X) \rightarrow \max \), \( y_2(X) \rightarrow \min \), \( y_3(X) \rightarrow \max \), \( f_\alpha(X) \rightarrow \min \). The parameters \( X = \{x_1, x_2\} \) vary within the following limits: \( 2.0 \leq x_1 \leq 3.5 \), \( 12 \leq x_2 \leq 30 \).

It is required: Build a model of the technological process in the form of a vector problem. Solve the vector problem with equivalent criteria. Choose a priority criterion. Set the numerical value of the priority criterion. Make the best (optimal) solution.

### 3.1.4.2 Digital Transformation of the Mathematical Model of the Technological Process

**Step 1. Construction of the mathematical model of the technological process in the conditions of a certainty.**

We form a vector problem of mathematical programming for which the criteria (104) and restrictions (105) on the process parameters are known:

\[
f_4(X) = -0.245 - 0.7470x_1 + 0.3832x_1^2 + 0.0442x_2 + 0.0012x_2^2 - 0.0346x_1 \cdot x_2 \quad (106)
\]

\[
\begin{align*}
2.0 & \leq x_1 \leq 3.5 \\
12.0 & \leq x_2 \leq 30.0
\end{align*}
\]

**Step 2. Digital transformation of mathematical model construction in conditions of uncertainty.**

Digital transformation consists in the use of qualitative and quantitative descriptions of the technological system obtained on the principle of “input-output” in Table 3. Converting information-initial data \( y_1(X), y_2(X), y_3(X) \) into a functional form \( f_1(X), f_2(X), f_3(X) \) is carried out by using mathematical methods of regression analysis. The initial data of Table 3 are formed as a matrix \( I \) in the MATLAB system:

\[
I = [X, Y] = \{x_{i1}, x_{i2}, y_{i1}, y_{i2}, y_{i3}, i = 1, M\} \quad (107)
\]

For each experimental data set \( y_k, k = 1, \overline{3} \) the regression function is constructed using the least squares method \( \min \Sigma_{i=1}^{M} (y_i - \overline{y})^2 \) in the MATLAB system. For this, the polynomial \( A \) is formed, which determines the interrelation of the parameters \( X = \{x_{i1}, x_{i2}\} \) and the function \( \overline{y}_k = f(X, A_k) \), \( k = 1, \overline{3} \). The result is a system of coefficients \( A_k = \{ A_{0k}, A_{1k}, \ldots, A_{uk}\} \), which determine the coefficients of the polynomial (function):

\[
f_k(X, A) = a_{0k} + a_{1k}x_1 + a_{2k}x_1^2 + a_{3k}x_2 + a_{4k}x_2^2 + a_{5k}x_1 \cdot x_2, \quad k = 1, \overline{3} \quad (108)
\]

The experimental data \( I_k = \{x_{i1}, x_{i2}, y_{i}\} \) of the matrix \( I_1 = [X, Y] \) (107) are converted into the function \( f_i(X) \) in the form (107), which, taking into account the obtained coefficients, takes the form:
\[ f_1(X) = 11.47 - 4.899x_1 + 0.8868x_1^2 - 0.003x_2 + 0.0048x_2^2 - 0.0595x_1x_2 \] 

The experimental data \( I_2 = \{x_{1i}, x_{2i}, y_{2i}\} \) of the matrix \( I = [X, Y] \) (107) are converted into the function \( f_2(X) \) in the form (97), which, taking into account the obtained coefficients:

\[ f_2(X) = 8.817 - 7.6809x_1 + 2.1456x_1^2 + 0.1851x_2 + 0.0894x_2^2 - 0.1454x_1x_2. \] 

The experimental data \( I_3 = \{x_{1i}, x_{2i}, y_{3i}\} \) of the matrix \( I = [X, Y] \) (107) are converted to the function \( f_3(X) \) in the form (97), which, taking into account the obtained coefficients:

\[ f_3(X) = -0.1225 - 0.3735x_1 + 0.1916x_1^2 + 0.0221x_2 + 0.0006x_2^2 - 0.0173x_1x_2 \] 

**Step 3. Construction of a mathematical model of the technological process. General part for the conditions of certainty and uncertainty.**

To build a mathematical model of a technological process, we use: the functions obtained by the conditions of certainty Equation (106) and uncertainty Equations (109)-(111). All criteria are \( K_1 = 2 \) at maximization: \( f_1(X) \rightarrow \text{max}, f_3(X) \rightarrow \text{max} \) and \( K_2 = 2 \) at minimization: \( f_2(X) \rightarrow \text{min}, f_4(X) \rightarrow \text{min}. K = K_1 \cup K_2 \).

As a result, the model of the functioning of the technological process will be represented by the vector problem of mathematical programming - VPMP:

\[
\begin{align*}
\text{opt} \quad F(X) &= \{\max f_1(X) = 11.470 - 4.8992x_1 + 0.8868x_1^2 - 0.0030x_2 + 0.0048x_2^2 - 0.0595x_1x_2 \} \\
\text{max} f_3(X) &= -0.122 - 0.3735x_1 + 0.1916x_1^2 + 0.0221x_2 + 0.0006x_2^2 - 0.0173x_1x_2 \\
\text{min} f_2(X) &= \{\min f_2(X) = 8.8176 - 7.6809x_1 + 2.1456x_1^2 + 0.1851x_2 + 0.0894x_2^2 - 0.1454x_1x_2 \} \\
\text{min} f_4(X) &= -0.245 - 0.747x_1 + 0.383x_1^2 + 0.044x_2 + 0.0012x_2^2 - 0.0346x_1x_2 \\
\end{align*}
\]

At restrictions:

\[
\begin{align*}
2.0 \leq x_1 &\leq 3.5 \\
12.0 \leq x_2 &\leq 30.0
\end{align*}
\]

The vector problem of mathematical programming (112)-(116) represents a mathematical model of making optimal decisions in conditions of certainty and uncertainty in the aggregate, as a result of digital transformation.

**3.1.4.3 Stage 3: The Solution of the VPMP-The Model of the Technological Process**

To solve the vector problem of mathematical programming (112)-(116), we use the solution method based on the normalization of criteria and the principle of guaranteed results, presented in section 3.3. The solution of the VPMP (112)-(116) with equivalent criteria is represented as a sequence of steps.

**Step 1. Solving VPMP (112)-(116) for each criterion separately, using the fmincon(...) function of the MATLAB system[19], the call to the fmincon(...) function is considered in the study[15].**

As a result of the calculation for each criterion, we obtain optimum points: \( X_k^* \) and \( f_k^* = f_k(X_k^*) \), \( k = 1, K, K = 4 \) is the values of the criteria at this point, that is, the best solution for each criterion:

\[
\begin{align*}
X_1^* &= \{x_1 = 2, x_2 = 30\} \\
X_2^* &= \{x_1 = 3.5, x_2 = 30\} \\
X_3^* &= \{x_1 = 3.5, x_2 = 12.0\} \\
X_4^* &= \{x_1 = 2.9, x_2 = 12.0\}
\end{align*}
\]

\[
\begin{align*}
f_1^* &= f_1(X_1^*) = -5.8833 \\
f_2^* &= f_2(X_2^*) = -78.9641 \\
f_3^* &= f_3(X_3^*) = -0.54235 \\
f_4^* &= f_4(X_4^*) = -0.3334
\end{align*}
\]
The constraints and optimum points $X^*_1, X^*_2, X^*_3, X^*_4$ (115) in the coordinates $\{x_1, x_2\}$ are shown in Figure 7A. The set of valid points $S$ is not empty and is a compact:

$$S = \{X \in \mathbb{R}^N | 2.0 \leq x_1 \leq 3.5, 12.0 \leq x_2 \leq 30.0\} \neq \emptyset.$$ (119)

The set of points that are Pareto optimal, $S^p$ represents the area of the set of points that lie between the points of the optimum $X^*_1, X^*_2, X^*_3, X^*_4$. We see that in this VPMP, the set of admissible points $S$ and the set of points optimal in Pareto, $S^p$, are equal to each other: $S = S^p$.

**Step 2. The worst unchanging part of each criterion (anti-optimum) is determined, a superscript zero.**

$$\begin{align*}
X^0_1 &= \{x_1 = 3.50, x_2 = 22.0052\} \\
X^0_2 &= \{x_1 = 2.1952, x_2 = 12.0\} \\
X^0_3 &= \{x_1 = 2.0, x_2 = 12.0\} \\
X^0_4 &= \{x_1 = 3.5, x_2 = 12.0\}
\end{align*}$$ (120)

$$\begin{align*}
f^0_1 &= f_1(X^0_1) = 2.8663 \\
f^0_2 &= f_2(X^0_2) = 13.56 \\
f^0_3 &= f_3(X^0_3) = -0.1667 \\
f^0_4 &= f_4(X^0_4) = -1.0847
\end{align*}$$ (121)

**Step 3. A systematic analysis of the set of Pareto optimal points (i.e., analysis by each criterion) is performed.** At the optimum points $X^* = \{X^*_1, X^*_2, X^*_3, X^*_4\}$, the values of the objective functions $F(X^*) = \|f_q(x^*_1)\|^k_{q=1}^{\infty}$ are determined, the vector $D = (d_1 d_2 d_3 d_4)^T$ deviations for each criterion on the admissible set $S$:

$$d_k = f^*_k - f^0_k, k = 1, 4,$$

$$F(X^*) = \begin{bmatrix}
-5.8833 & -79.3272 & -0.0619 & 0.1238 \\
-3.1731 & -78.9641 & -0.3038 & 0.6077 \\
-3.3468 & -17.2060 & -0.5423 & 1.0847 \\
-4.4505 & -13.6434 & -0.1667 & -0.3334
\end{bmatrix}$$ (122)

$$d_k = \begin{bmatrix}
3.0170 \\
65.4035 \\
0.7090 \\
1.4181
\end{bmatrix}$$ (123)

and the relative estimation matrix:

$$\lambda(X^*) = \left[\frac{\lambda_k(X^*_q)}{d_k} \right]_{q=1}^{\infty}$$

$$\lambda_k(X) = (f^*_k - f^0_k)/d_k$$

$$\lambda(X^*) = \begin{bmatrix}
1.0000 & 1.0056 & 0.3224 & 0.6776 \\
0.1017 & 1.0000 & 0.6636 & 0.3364 \\
0.1593 & 0.0557 & 1.0000 & 0.0 \\
0.5251 & 0.0013 & 0.0 & 1.0000
\end{bmatrix}$$ (124)

Where

$$X^* = \{X^*_1, X^*_2, X^*_3, X^*_4\}$$

The analysis of the values of criteria Equations (122)-(123) in relative estimates Equation (124) showed that at the points of optimum $X^*$ (diagonally) the relative estimate is equal to one. The remaining criteria are significantly less than one. It is required to find a parameter (point) at which the relative estimates are closest to unity. The solution of this problem is aimed at solving the $\lambda$-problem - step 4.
Figure 7. The schematic diagrams of set of admissible points and solution of the \( \lambda \)-problem on Theory of Vector Optimization. A: The set of admissible points \( S \) and Pareto optimal points \( S^0 \in S; X_1^*, X_2^*, X_3^*, X_4^*; X^0 \) in a two-dimensional system of coordinates \( \{x_1, x_2\} \); B: Geometric interpretation of the solution of the \( \lambda \)-problem coordinate \( \{x_1, x_2\} \) and \( \lambda \); C: Result the solution of the problem (99)-(103) in the three-dimensional coordinate system \( \{x_1, x_2, \lambda\} \); D: Geometric interpretation of symmetry in modeling of technical system with normalized criteria: \( \lambda_1(X), ..., \lambda_4(X) \).

**Step 4.** The construction of the \( \lambda \)-problem is carried out in two stages: a maximin optimization problem with normalized criteria is initially constructed

\[
\lambda^* = \max_{X \leq 0} \min_{k \in K} \lambda_k(X), \ G(X) \leq 0, \ X \geq 0 \quad (13)
\]

At the second stage, the maximin problem (119) is transformed into a standard mathematical programming problem (\( \lambda \)-problem):

\[
\lambda^* = \max \lambda \quad (19)
\]

At restrictions
\[
\begin{align*}
\lambda & = -11.47 - 4.8992 \times x_1 + 0.8868 \times x_1^2 - 0.0030 \times x_2 + 0.0048 \times x_2^2 - 0.0595 + x_1 \times \frac{f_1}{f_1^2} - f_1 \leq 0 \\
\lambda & = -0.1225 - 0.3735 \times x_1 + 0.1916 \times x_1^2 + 0.221 \times x_2 + 0.0067 \times x_2^2 + 0.0173 + x_1 \times \frac{f_1}{f_1^2} - f_1 \leq 0 \\
\lambda & = 8.8176 + 7.6809 \times x_1 - 2.1456 \times x_1^2 + 0.1851 \times x_2 + 0.0894 \times x_2^2 + 0.1454 + x_1 \times \frac{f_1}{f_1^2} - f_1 \leq 0 \\
\lambda & = -0.2450 - 0.7470 \times x_1 + 0.3832 \times x_1^2 + 0.0442 \times x_2 + 0.001245 \times x_2^2 + 0.0345 + x_1 \times \frac{f_1}{f_1^2} - f_1 \leq 0 \\
\end{align*}
\]

(125)

Where the vector of unknowns has dimension of \(N+1: X = \{x_1, ..., x_N, \lambda \}, N=2\).

**Step 5. Solution of the \(\lambda\)-problem. For this purpose we use the function fmincon(…)[19]:**

\([X_0, L_0] = \text{fmincon}(\text{'ZTS2Krit'}, X_0, A_0, b_0, Aeq, beq, lb_0, ub_0, \text{'ZTS2LConst'}, \text{options})\)

As a result of the solution of VPMP (99)-(103) at equivalent criteria and \(\lambda\)-problem corresponding to:

\[
X^o = \{X^o, \lambda^o\} = \{x_1 = 2.8039, x_2 = 30.0, \lambda^o = 0.3541\} \quad (127)
\]

shown in Figure 7B:

\[
f_k(X^o), k = \overline{1, K} \text{ - values of criteria (characteristics of the technological process), } \lambda_k(X^o), k = \overline{1, K} \text{ - values of relative estimates:}
\]

\[
F(X^o) = \{f_1(X^o) = 3.934, f_2(X^o) = 77.93, f_3(X^o) = 0.0844, f_4(X^o) = 0.1687\} \quad (128)
\]

\[
\lambda(X^o) = \{\lambda_1(X^o) = 0.3541, \lambda_2(X^o) = 0.984, \lambda_3(X^o) = 0.354, \lambda_4(X^o) = 0.6459\} \quad (129)
\]

\[
\lambda^o = \min \{\lambda_1(X^o), \lambda_2(X^o), \lambda_3(X^o), \lambda_4(X^o)\} = 0.3541 \quad (130)
\]

\(\lambda^o = 0.3541\) is the maximum lower level among all relative estimates, measured in the relative units.

\(\lambda^o\) - also called a guaranteed result in the relative units: \(\lambda_k(X^o) \forall k \in K\), and, accordingly, the characteristics of the technological process \(f_k(X^o) \forall k \in K\) it is impossible to improve, without worsening at the same time other characteristics. Note that, in accordance with theorem 1, at point \(X^o\) (127), criteria 1 and 3 are contradictory. This contradiction is determined by the equality \(\lambda_1(X^o) = \lambda_3(X^o) = \lambda^o = 0.3541\), and the remaining criteria by the inequality \(\lambda_2(X^o) = 0.9842, \lambda_4(X^o) = 0.6459 > \lambda^o\). Theorem 1 serves as the basis for determining the correctness of the solution of the vector problem of mathematical programming.

In the VPMP, as a rule, for two criteria, equality is satisfied:

\[
\lambda^o = \lambda_q(X^o) = \lambda_p(X^o), q, p \in K, X \in S
\]

and for other criteria it is defined as inequality: \(\lambda_2(X^o) > \lambda^o\).

### 3.1.4.4 Stage 4: Creation of Geometrical Interpretation of Results of the Decision in a Three-dimensional Coordinate

In the allowable set of points \(S\) formed by constraints (103), the optimum points \(X^* = \{X^*_1, X^*_2, X^*_3, X^*_4\}\), which are shown in Figure 7A, combined into a contour, represent the set of Pareto optimal points, \(S^o \subset S\). The coordinates of these points, as well as the characteristics of the technological process in relative units \(\lambda(X), \lambda(X), \lambda(X), \lambda(X)\) are shown in Figure 7B in the three-dimensional space \(\{x_1, x_2\}\) and where the third axis is the relative estimate \(\lambda\).

### 3.1.4.5 Stage 5: Decision-making in the Technological Process Model at the Set Priority of Criterion

To solve VPMP (112)-(116) methods are presented, based on axiomatic normalization of criteria and principle of guaranteed result, as well as axiomatic priority of criterion, resulting from axiom 2, 3 and principle of optimality 2, which are presented in section 3.2[19].

The decision maker is usually the process designer.
Step 1. We solve a VPMP (112)-(116) with equivalent criteria.

The algorithm of the decision is presented in Stage 3. Numerical results of the solution of the VPMP are given above. Pareto's great number of $S_i \subseteq S$ lies between optimum points $X_1^*, X_2^*, X_3^*, X_4^*, X_5^*$. This information is the basis for further research on the structure of the Pareto set. The decision maker is usually the technological process designer. If results of the solution of a VPMP with equivalent criteria do not satisfy the person making the decision, then the choice of an optimal solution is carried out from any subset of points of $S_1^*, S_2^*, S_3^*, S_4^*$.

Step 2. Choice of priority criterion of $q \in K$.

From the theory (Theorem 2) it is known that in an optimum point of $X^o$ there are always two most contradictory criteria: $q \in K$ and $v \in K$ for which in the relative unit's precise equality is carried out: 

$$
\lambda^o = \lambda_q(X^o) = \lambda_p(X^o), \quad q, p \in K, X \in S
$$

and for the others it is carried out inequalities:

$$
\lambda^o \leq \lambda_k(X^o) \forall k \in K, q \neq v \neq k
$$

On the display the message is given:

$q=$input ('Enter priority criterion (number) of $q=$') - Entered: $q=3$.

For the choice of the priority criterion on the display the message about results of the solution of $\lambda$-problem in physical and relative units is given:

Criteria (119) in $X^o$ optimum point:

$$
F(X^o) = \{f_1(X^o) = 3.9345, f_2(X^o) = 77.9319, f_3(X^o) = 0.0844, f_4(X^o) = 0.1687\}
$$

The relative estimates (120) in $X^o$:

$$
\lambda(X^o) = \{\lambda_1(X^o) = 0.3541, \lambda_2(X^o) = 0.9842, \lambda_3(X^o) = 0.3541, \lambda_4(X^o) = 0.6459\}
$$

From the function $\lambda(X^o)$ it is clear that the first and the third are the most contradictory criteria:

$$
\lambda^o = \{\lambda_1(X^o), \lambda_3(X^o)\} = 0.3541
$$

Select from Figure 7B the first and third criteria and present (120) $\{\lambda_1(X^o), \lambda_3(X^o)\}$ in the relative units in Figure 7C. From a pair of the contradictory criteria, a criterion chosen by the decision maker would be improved. Such a criterion is called "priority criterion", we will designate it $q = 3 \in K$. This criterion is investigated in interaction with the first criterion of $k = 1 \in K$.

Step 3. Numerical limits of the change of the size of a priority of the criterion of $q = 3 \in K$ are defined.

For priority criterion of $q = 3 \in K$ changes of the numerical limits in the physical units upon transition from $X^o$ optimum point to the point of $X_q^*$ received on the first step at equivalent criteria are defined. $q=3$ given about criterion are given for the screen:

$$
f_q(X^o) = -0.084354 \leq f_q(X) \leq -0.54235 = f_q(X_q^*), q = 3 \in K, (131)
$$

In the relative units the criterion of $q=3$ changes in the following limits:

$$
\lambda_q(X^o) = 0.35407 \leq \lambda_q(X) \leq 1 = \lambda_q(X_q^*), q = 3 \in K
$$

Step 4. Choice of size of the priority criterion of $q \in K$.

On the message: "Enter the size of priority criterion $f_q^{eq}$ - we enter, the size of the characteristic $q = 3 \in K$: $f_q = 0.3$

Step 5. The relative assessment is calculated.

For the chosen size of priority criterion $f_q=0.3$ the relative assessment is calculated:

$$
\lambda_q = \frac{f_q - f_q^0}{f_q^* - f_q^0} = \frac{0.3 - (-0.1617)}{0.5423 - (-0.1617)} = 0.6582
$$

which upon transition from $X^o$ point to $X_q^*$ lies in the limits:
Step 6. Calculate the coefficient of the linear approximation.
Assuming the linear nature of the change of the criterion of \( f_q(X) \) in (6.69) and according to the relative assessment of \( \lambda_q \), using standard linear approximation techniques, we will calculate the proportionality coefficient between \( \lambda_q(X^p) \), \( \lambda_q \) which we will call \( \rho \):

\[
\rho = \frac{\lambda_q - \lambda_q(X^p)}{\lambda_q(X^p) - \lambda_q(X^o)} = \frac{0.6582 - 0.35407}{1 - 0.35407} = 0.4708, \quad q = 3 \in K
\]

Step 7. Let's calculate coordinates of a priority of the criteria with dimension of \( f_q \).
Assuming the linear nature of change of a vector of \( X^q = \{x_1, x_2\} \), \( q = 3 \) we will determine point coordinates with dimension of \( f_q = 0.3 \), the relative assessment \( \lambda_q \):

\[
x_{\lambda=0.7} = \{x_1 = X^o(1) + p(X^o(1) - X^o(2)), x_2 = X^o(2) + p(X^o(2) - X^o(2))\} \tag{132}
\]

Where

\[
\{X^o = \{X^o(1) = 2.8039, X^o(2) = 30.0\}, X^o(1) = 80.0, X^o(2) = 0.0\}
\]

As result of the decision (122) we will receive \( X^p \) point with coordinates:

\[
X^p = \{x_1 = 3.1317, x_2 = 21.525\}
\]

Step 8. Calculation of the main indexes of a point of \( X^p \).
For the received \( X^p \) point, we will calculate:

all criteria in the physical units:

\[
f_k(X^p) = \{f_k(X^p), k = 1, K\}
\]

\[
f(X^p) = \{f_1(X^p) = 2.9775, f_2(X^p) = 41.4096, f_3(X^p) = 0.1744, f_4(X^p) = 0.3489\}
\]

All relative estimates with the criterion priority:

\[
\lambda_q = \{\lambda_q, k = 1, K\}, \lambda_k(X^p) = \frac{f_k(X^p) - f_k^q}{f_k^q}, k = 1, K
\]

\[
\lambda_k(X^p) = \{\lambda_1(X^p) = 0.0369, \lambda_2(X^p) = 0.4258, \lambda_3(X^p) = 0.4811, \lambda_4(X^p) = 0.5189\}
\]

minimum relative assessment: \( \min(\lambda_k(X^p)) = 0.0369 \);
vector of priorities of the third criterion over other criteria:

\[
p^q(X) = \{p_k^q = \frac{\lambda_q(X^p)}{\lambda_k(X^p)}, k = 1, K\}
\]

\[
p^q = [p_1^q = 6.2789, p_2^q = 17.94, p_3^q = 1.0, p_4^q = 0.9]
\]

Any point from Pareto's set \( X^o = \{X^o, X^o\} \in S^o \) can be similarly calculated.

Step 9. Analysis of the results of the solution of the process model represented by the vector problem from the current of symmetry.
In the considered vector problem of nonlinear programming (101)-(105) with four heterogeneous criteria - the model of the technological system and the corresponding numerical variant (101)-(105) and built on its basis \( \lambda \)-problems (115) - (116) we obtained the optimum point \( X^o \) and the maximum relative estimate \( \lambda^o \):

\[
X^o = \{X^o, \lambda^o\} = \{X^o = \{x_1 = 2.8039, x_2 = 30.0, \lambda^o = 0.3541\}\}
\]

At the optimum point \( X^o \) the criteria in natural units are:

\[
F(X^o) = \{f_1(X^o) = 3.9345, f_2(X^o) = 77.9319, f_3(X^o) = 0.0844, f_4(X^o) = 0.1687\}
\]

and relative units:

\[
\lambda(X^o) = \{\lambda_1(X^o) = 0.3541, \lambda_2(X^o) = 0.9842, \lambda_3(X^o) = 0.3541, \lambda_4(X^o) = 0.6459\}
\]
This result confirms the proofs of theorem 1 about the most contradictory criteria in the vector optimization problem, i.e.,

$$\lambda^o = \lambda_1(X^o) = \lambda_3(X^o) = 0.3541, \{1, 3\} \in K, X^o \in S$$

The rest of the criteria in relative units lie within:

$$\{\lambda_2(X^o) = 0.9842, \lambda_4(X^o) = 0.6459\} > \lambda^o$$

Theoretically, the point $X^o$ is the center of symmetry. Indeed, the $X^o$ point lies between the points of the optimum $X_1^1$ and $X_3^3$ obtained for each criterion (step 1), in which $\lambda_1(X_1^1) = 1$, $\lambda_3(X_3^3) = 1$. Let’s show geometrically symmetry in Figure 7D.

For the two criteria of the first $\lambda_1(X)$ and the third $\lambda_3(X)$, this example considers an even numerical symmetry.

Analysis of results in the VPMP. The calculated size of criterion $f_q(X_i^o)$, $q \in K$ is usually not equal to the set $f_q$. The error of the choice of $\Delta f_q = |f_q(X_i^o) - f_q| = |0.1744 - 0.3| = 0.125$ is defined by an error of linear approximation, $\Delta f_q\% = \frac{\Delta f}{f_q} \times 100 = 40.2\%$.

In the course of the modeling parametric restrictions (6.53) can be changed, i.e., some set of optimum decisions is received. Choose a final version which in our example included from this set of optimum decisions:

(1) Parameters of technological process $X^o = \{x_1 = 2.8039, x_2 = 30.0\}$;
(2) The parameters of the technological system at a given priority criterion $q = 3$: $X^q = \{x_1 = 3.1317, x_2 = 21.5248\}$.

3.1.4.6 Stage 6: Geometrical Interpretation of Results of the Decision in the Technological Process in Three to a Measured Frame in Physical Units

We represent these parameters in a two-dimensional $x_1, x_2$ and three dimensional coordinate system $x_1, x_2$ and $\lambda$ in Figures 7A-7C, and also in physical units for each function $f_1(X), f_2(X), f_3(X)$ and $f_4(X)$ on Figures 8A-6D respectively.

![Object](https://example.com/object.jpg)
In the aggregate, the first option presented:
(1) point \( X^0 = \{ x_1, x_2 \} \);
(2) the functional characteristics \( F(X^0) = \{ f_1(X^0), f_2(X^0), f_3(X^0), f_4(X^0) \} \)
(3) relative estimates of \( \lambda(X^0) = \{ \lambda_1(X^0), \lambda_2(X^0), \lambda_3(X^0), \lambda_4(X^0) \} \);
(4) maximin \( \lambda^0 \) - relative level such that \( \lambda^0 \leq \lambda_k(X^0) \forall k \in K \).

There is an optimal solution with equivalent criteria (characteristics), and the procedure for obtaining is the adoption of the optimal solution in the process with equivalent criteria (characteristics).

The second option:
(1) point \( X^q = \{ X^{1q}, X^{2q} \} \); characteristics \( f(X^q) = \{ f_1(X^q), f_2(X^q), f_3(X^q), f_4(X^q) \} \);
(2) relative estimates \( \lambda_k(X^q) = \{ \lambda_1(X^q), \lambda_2(X^q), \lambda_3(X^q), \lambda_4(X^q) \} \);
(3) maximin \( \lambda^{eq} \) is a relative level such that \( \lambda^{eq} \leq \lambda_1(X^q), \lambda_2(X^q), \lambda_3(X^q), \lambda_4(X^q) \);

There is an optimal solution with the priority of the third characteristics (criterion) relative to other criteria. The procedure for obtaining the point \( X^q \) is the adoption of the optimal solution for a given priority of the criterion.

The vector optimization theory, methods for solving vector problems of mathematical programming with equivalent criteria and a given criterion priority allow you to select any point from the set of Pareto optimal points and show the optimality of this point.

### 3.1.5 Modeling and Selection of Optimal Parameters of a Structure of Material Under Conditions of Certainty and Uncertainty Based on Theory of Vector Optimization

The problem of a decision making of structure of material about which are known is considered: first, given about the functional interrelation of several characteristics with its components which are a part of this material - the conditions of definiteness are absent; secondly, data on some set of discrete values of several characteristics - the experimental results, in interrelation with discrete values of the components which are a part of this material - uncertainty conditions.

The first stage, as well as the stage of analyzing the result of the solution and selecting the priority criterion and its value, is performed by the material designer. The remaining stages are performed by a mathematician-programmer.

#### 3.1.5.1 Stage 1: The Technical Assignment: The Choice of the Optimal Parameters of the Structure of Material

The choice of optimum parameters of material is carried out by the designer of material. Material which structure is defined by four components: \( Y = \{ y_1, y_2, y_3, y_4 \} \) - a vector (operated) variable. Input data for a decision making are four characteristics:

\[
H(Y) = \{ h_1(Y), h_2(Y), h_3(Y), h_4(Y) \} \quad (133)
\]

Conditions of certainty are absent. For conditions of uncertainty, four characteristics are known discrete values of components (experimental) \( Y = \{ y_1, y_2, y_3, y_4 \} \) - with the corresponding discrete values of characteristics of \( H(Y) = \{ h_1(Y), h_2(Y), h_3(Y), h_4(Y) \} \). Numerical values of parameters \( Y \) and characteristics of \( H(X) \) are presented in Table 4.

<table>
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<th>( y_1 )</th>
<th>( y_2 )</th>
<th>( y_3 )</th>
<th>( y_4 )</th>
<th>( h_1(Y) )</th>
<th>( h_2(Y) )</th>
<th>( h_3(Y) )</th>
<th>( h_4(Y) )</th>
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Parametric restrictions change in the following limits (as a percentage):

$$\begin{align*}
21 & \leq y_1 \leq 79 \\
5 & \leq y_2 \leq 59 \\
2.1 & \leq y_3 \leq 9 \\
2.2 & \leq y_4 \leq 7 \\
\end{align*}$$

(134)

On the parameters $Y = \{y_1, y_2, y_3, y_4\}$ are in total imposed restriction:

$$\begin{align*}
y_1 + y_2 + y_3 + y_4 &= 100 \\
y_1 &\in [20.50.80.] \\
y_2 &\in [0.30.60.] \\
y_3, y_4 &\in [2.5.8.] \\
\end{align*}$$

(135)
In the made decision, assessment size on the first and third characteristic it is desirable (criterion), to receive as above (max): \( y_1(X) \rightarrow \text{max} \), \( y_3(X) \rightarrow \text{max} \), on second and fourth as low as possible is possible (min)\(^2\) \( y_2(X) \rightarrow \text{min} \), \( f_4(X) \rightarrow \text{min} \). In the pilot studies the rate of an increase of parameters \( Y \) will be defined in the following limits:

1. To construct a mathematical model of the structure of the studied material in the form of a VPMP.
2. To carry out model operation: first, on the basis of the constructed mathematical model, secondly, on the basis of methods of solution of a VPMP of non-linear programming at equivalent criteria, and, thirdly, the software developed for these purposes in the MATLAB system.
3. To make an optimal solution: The choice of optimum structure (composition) of material according to its functional characteristics taking into account their equivalence.
4. To choose the optimum composition of structure of material according to its functional characteristics taking into account a priority of the third criterion. The size of the third criterion \( h_3(X^9) = 138.2 \) is received at model operation of the technical system.

### 3.1.5.2 Stage 2: Construction of a Mathematical Model of the Structure of Material

It is carried out by the mathematician - the programmer. At a stage of the choice of priority criterion and its size it is carried out by the designer of material.

**Step 1. Construction in the conditions of a certainty.**

The step 1 is not carried out as characteristics of material in the conditions of certainty are not set.

**Step 2. Construction of a mathematical model in the form VPMP of an uncertainty.**

Construction of a mathematical model in the conditions of uncertainty consists in use of qualitative, quantitative descriptions of material by the principle "input-output" to Table 4. Using methods of the regression analysis, input data of Table 4:

\[
H(Y) = \{ h_1(Y), h_2(Y), h_3(Y), h_4(Y) \} \quad (133)
\]

will be transformed to the functional type of

\[
F(Y) = \{ f_1(Y), f_2(Y), f_3(Y), f_4(Y) \} \quad (137)
\]

**Step 3. Construction of a mathematical model of the structure of material.**

For creation of a mathematical model of material, we use results of the regression analysis. We consider the turned-out functions \( f_1(Y), f_2(Y), f_3(Y), f_4(Y) \) as criterion of a vector problem. This criterion determines the purposefulness of characteristics of material. A set of criteria \( K = 4 \) include two criteria \( f_1(Y) \to \text{max}, f_2(Y) \to \text{max} \) and two criteria of \( f_2(Y) \to \text{min}, f_4(Y) \to \text{min} \).

As a result, model of functioning is represented by the vector problem of mathematical programming:

\[
\text{opt} \; F(Y) = \{ \text{max} \; F_1(Y), \; \text{max} \; F_1(Y) \} = 323.8 - 1.875y_1 - 2.911y_2 + 8.939y_3 + 10.94y_4 + 0.0673y_1y_2 - 0.0431y_1y_3 - 0.1176y_1y_4 + 0.0516y_2y_3 - 0.0723y_2y_4 + 0.0021y_3y_4 + 0.0999y_1^2 + 0.0081y_2^2 - 0.169y_3^2 - 0.349y_4^2 \quad (138)
\]

\[
\text{max} \; f_3(Y) = 95.71 + 0.6598y_1 + 0.4493y_2 - 0.3094y_3 - 1.833y_4 - 0.091y_1y_2 - 0.0057y_1y_3 + 0.0134y_1y_4 - 0.0119y_2y_3 + 0.0003y_3y_4 - 0.0002y_1^2 - 0.0002y_2^2 + 0.0233y_3^2 + 0.086y_4^2 \quad (139)
\]

\[
\text{min} \; F_2(Y) = \{ \text{min} \; f_2(Y) \} = 95.48 + 23.9y_1 + 30.86y_2 - 25.85y_3 - 45y_4 - 0.64y_1y_2 + 0.3919y_1y_3 + 0.6227y_1y_4 - 0.1069y_2y_3 + 0.2722y_2y_4 - 0.0078y_3y_4 + 0.0332y_1^2 + 0.0304y_2^2 + 2.67y_3^2 + 2.205y_4^2 \quad (140)
\]

\[
\text{min} \; f_a(Y) = 21.0 - 0.0081y_1 - 0.7y_2 - 0.3605y_3 + 0.9769y_4 + 0.0115y_1y_2 + 0.059y_1y_3 - 0.0001y_1y_4 + 0.0363y_2y_3 + 0.0002y_2y_4 + 0.0003y_3y_4 - 0.015y_1^2 + 0.0025y_2^2 - 0.042y_3^2 - 0.0216y_4^2 \quad (141)
\]

At restrictions:

\[
y_1 + y_2 + y_3 + y_4 = 100
\]

\[
\begin{cases}
21 \leq y_1 \leq 79 \\
5 \leq y_2 \leq 59 \\
2.1 \leq y_3 \leq 9 \\
2.2 \leq y_4 \leq 7
\end{cases}
\]
The vector problem of mathematical programming (138)-(142) represents the mathematical model of optimal decision making, i.e., the choice of the optimal structure of the material in conditions of certainty and uncertainty in the total.

3.1.5.3 Stage 3: The Solution of a Vector Problem of Mathematical Programming - mathematical Model of Material.

Algorithm 1. The solution of a vector task with equivalent criteria.

The solution of a vector problem of mathematical programming (138)-(142) with equivalent criteria can be represented as a sequence of steps.

Step 1. Decides VPMP (125)-(130) by each criterion separately, at the same time the function \textit{fmincon} (...) of the MATLAB system is used, the appeal to the function \textit{fmincon} (...) is considered in the study\textsuperscript{[160]}. As a result of calculation for each criterion we receive optimum points: \(X_k^*\) and \(f_k^* = f_k(X_k^*), k = 1, K \ K = 4\) sizes of criteria in this point, i.e., The solution of a problem of non-linear programming in three-dimensional frames of \(x_1, x_3\) and \(f(x)\) was presented on Figure 9A.

The best decision on each criterion:

\[
\begin{align*}
X_1^* & = \{x_1 = 46.57, x_2 = 42.23, x_3 = 8, x_4 = 2.2\} \\
X_2^* & = \{x_1 = 50.5, x_2 = 45.2, x_3 = 2.1, x_4 = 2.2\} \\
X_3^* & = \{x_1 = 79, x_2 = 11.9, x_3 = 2.1, x_4 = 7\} \\
X_4^* & = \{x_1 = 36.7, x_2 = 59, x_3 = 2.1, x_4 = 2.2\} \\
\end{align*}
\]

\[
\begin{align*}
f_1^* & = f_1(X_1^*) = 387.99 \\
f_2^* & = f_2(X_2^*) = 2215.91 \\
f_3^* & = f_3(X_3^*) = 150.2 \\
f_4^* & = f_4(X_4^*) = 30.71 \\
\end{align*}
\]

The location of the optimum points \(X_1^*, X_2^*, X_3^*, X_4^*\) in the region of the constraints (129)-(130) in the coordinates \(\{x_1, x_3\}\) was shown in Figure 9A. The set of points of \(S^0\) lying in the domain of restrictions between the points \(X_1^*, X_2^*, X_3^*, X_4^*\) represent a set of Pareto optimal points.

Step 2. The worst unchangeable part of each criterion is defined (anti-optimum). (A superscript zero)

\[
\begin{align*}
X_1^0 & = \{x_1 = 31.9, x_2 = 59, x_3 = 2.1, x_4 = 7\} \\
X_2^0 & = \{x_1 = 79.0, x_2 = 6.0, x_3 = 8.0, x_4 = 7\} \\
X_3^0 & = \{x_1 = 31.9, x_2 = 57.43, x_3 = 8.0, x_4 = 2.666\} \\
X_4^0 & = \{x_1 = 62.71, x_2 = 22.89, x_3 = 8.0, x_4 = 6.399\} \\
\end{align*}
\]

\[
\begin{align*}
f_1^0 & = f_1(X_1^0) = 296.6 \\
f_2^0 & = f_2(X_2^0) = -3903.1 \\
f_3^0 & = f_3(X_3^0) = 114.87 \\
f_4^0 & = f_4(X_4^0) = 73.63 \\
\end{align*}
\]

The obtained points of the anti-optimum \(X_1^0, X_2^0, X_3^0, X_4^0\) are shown in Figure 10A-10D respectively.
Figure 9. The schematic diagrams of set of admissible points and solution of the $\lambda$-problem on Theory of Vector Optimization. A: The set of admissible points $S$ and Pareto optimal points $S^0 \subset S$; $X_1, X_2, X_3, X_4; X^0$ in two-dimensional system of coordinates $(x_1, x_3)$; B: Geometric interpretation of the solution of the $\lambda$-problem coordinate $x_1, x_3$ and $\lambda$; C: Solution of the $\lambda$-problem; D: Geometric interpretation of symmetry in modeling of technical system with normalized criteria: $\lambda_1(X), ..., \lambda_4(X)$.

Step 3. Systems analysis of a set of points that are Pareto-optimal is performed, (i.e., the analysis by each criterion). In points of an optimum of $X^* = \{X_1^*, X_2^*, X_3^*, X_4^*\}$ sizes of target functions of $F(X^*) = \|f_q(X^*)\|_{q=1}^{\infty}$ a vector of $D = (d_1, d_2, d_3, d_4)^T$ of deviations are determined by each criterion on an admissible set of $S$: $d_k = f_k^0 - f_k^*$, $k = 1, 4$, and matrix of the relative estimates of

$$\lambda(X^*) = \|\lambda_q^*(X^*)\|_{q=1}^{\infty}$$

Where

$$\lambda_k(X) = (f_k^0 - f_k^*)/d_k \quad (95)$$

$$F(X^*) = \begin{bmatrix}
388.01 & 2401.2 & 117.5 & 68.5 \\
353.0 & 2215.9 & 124.9 & 37.5 \\
264.2 & 3445.7 & 150.2 & 34.0 \\
330.1 & 2408.9 & 122.7 & 30.7
\end{bmatrix} \quad (147)$$

$$d_k = \begin{bmatrix}
91.4 \\
-1725.0 \\
35.36 \\
-42.9
\end{bmatrix} \quad (148)$$

$$\lambda(X^*) = \begin{bmatrix}
1.0000 & 0.8926 & 0.0742 & 0.197 \\
0.6171 & 1.0000 & 0.2832 & 0.8429 \\
-0.359 & 0.2872 & 1.0000 & 0.9239 \\
0.3669 & 0.8881 & 0.2208 & 1.0000
\end{bmatrix} \quad (149)$$
The analysis of sizes of criteria in the relative estimates $\lambda(X') = \frac{\lambda_q(X'_k)}{\|X'_q\|_{\#_{k=1, K}}}^{k=1, K}$ showed that at the points of the optimum $X' = \{X'_1, X'2, X'_3, X'_4\}$ (on diagonal) the relative assessment is equal to the unit. For other criteria, there is much less unit. It is required to find such point (parameters) at which the relative estimates are closest to the unit. The solution of this problem is directed to the solution of $\lambda$-problem - step 4.

Step 4. Creation of the $\lambda$-problem is carried out in two stages: originally the maximum problem of optimization with the normalized criteria is under construction:

$$\lambda^o = \max_{X_{\#k}} \min_{k \in K} \lambda_k(X), \ \forall(X) \leq 0, X \geq 0 \ (13)$$

which at the second stage will be transformed to a reference problem of mathematical programming ($\lambda$-problem):

$$\lambda^o = \max \lambda \ (19)$$

at restrictions

$$\begin{cases} 
\lambda - \frac{f_1(Y)-f_2}{f_1-f_2} \leq 0 \\
\lambda - \frac{f_2(Y)-f_3}{f_2-f_3} \leq 0 \\
\lambda - \frac{f_3(Y)-f_4}{f_3-f_4} \leq 0 \\
\lambda - \frac{f_4(Y)-f_1}{f_4-f_1} \leq 0 \\
x_1 + x_2 + x_3 + x_4 = 100 \\
21 \leq x_1 \leq 79 \\
5 \leq x_2 \leq 59 \\
2.1 \leq x_3 \leq 9 \\
2.2 \leq x_4 \leq 7 
\end{cases} \ (39)$$

where the vector of unknowns has dimension of $N + 1$: $X = \{x_1, ..., x_N, \lambda\}, N = 4$.

Step 5. Solution of the $\lambda$-problem. For this purpose we use the function \texttt{fmincon(...)}[^16]: [Xo, Lo]: \texttt{fmincon('Z_Mater_4Krit_L', X0, A0, b0, aeq, beq, lbo, ubo, 'Z_Mater_LConst')}

As a result of the solution of vector problem of mathematical programming (122)-(127) at equivalent criteria and $\lambda$-problem corresponding to it (131)-(133) received: $X^o$ is an optimum point - design data of material, presented in Figure 9B and 9C:

$$X^o = \{X^o, \lambda^o\} = \{X^o = \{x_1 = 69.5, x_2 = 24.1, x_3 = 4.144, x_4 = 2.2, \lambda^o = 0.546\} \ (151)$$

1, 3 criteria in the coordinates $x_1, x_3$ and $f_k(X^o)$, $k = 1, K$ - sizes of criteria (characteristics of material):

$$F(X^o) = \{f_1(X^o) = 346.5, f_2(X^o) = 2693.0, f_3(X^o) = 134.2, f_4(X^o) = 50.2\} \ (152)$$

$\lambda_k(X^o), k = 1, K$ - sizes of the relative estimates

$$\lambda(X^o) = \{\lambda_1(X^o) = 0.5461, \lambda_2(X^o) = 0.72, \lambda_3(X^o) = 0.5461, \lambda_4(X^o) = 0.5461\} \ (153)$$

$\lambda^o = 0.5461$ is the maximum lower level among all relative estimates measured in the relative units:

$$\lambda^o = \min(\lambda_1(X^o), \lambda_2(X^o), \lambda_3(X^o), \lambda_4(X^o)) \ (154)$$

$\lambda^o$ also call the guaranteed result in the relative units, i.e., $\lambda_k(X^o)$ and according to the characteristic of the material $f_k(X^o)$ it is impossible to improve, without worsening at the same time other characteristics.

Let's notice that according to the theorem 2[^4], in $X^o$ point criteria $k = 1, 3$ and $k = 4$ are contradictory. This contradiction is defined by equality of

$$\lambda^o = \min(\lambda_1(X^o), \lambda_2(X^o), \lambda_3(X^o), \lambda_4(X^o)), \lambda^o = 0.546$$

and other criteria of inequality $\{\lambda_k(X^o)\} > \lambda^o$. 
3.1.5.4 Stage 4: Creation of Geometrical Interpretation of Results of the Decision in a Three-dimensional Coordinate System

In an admissible point set of $S$ formed by restrictions (126)-(127), optimum points $X_1, X_2, X_3, X_4$, united in a contour, present a point set, Pareto optimal, to $S^o \subseteq S$. Coordinates of these points and also characteristics of material in the relative units: $\lambda_1(X), \lambda_2(X), \lambda_3(X), \lambda_4(X)$ are shown in Figure 9B in three measuring space: $x_1, x_2, x_3$ and $\lambda$, where the third axis of $\lambda$ - the relative assessment.

Let's carry out the analysis of results of the solution of a vector problem (the analysis of the choice of an optimal solution at the equivalent criteria – characteristics), using geometrical interpretation. For this purpose, in Figures 9B and 9C we will connect points: $X_1, X_2, X_3, X_4$ with a point of $X^o$ which conditionally represents the center of a Pareto set. As a result, conditionally received four subsets of points of $X \in S^o \subseteq S, q = 1, 4$. The subset of $S^o_1 \subseteq S^o \subseteq S$ is characterized by the fact that the relative assessment of $\lambda_1 \geq \lambda_2, \lambda_3, \lambda_4$, i.e., in the field of $S^o_1$ first criterion has a priority over the others. Similar to $S^o_2, S^o_3, S^o_4$ - subsets of points where the second, third, fourth criterion has a priority over the others respectively. A point set, Pareto optimal we will designate $S^o = S^o_1 \cup S^o_2 \cup S^o_3 \cup S^o_4$.

Coordinates of all received points and the relative estimates are presented in two-dimensional space in Figure 9A. These coordinates are shown in three measuring space $\{x_1, x_2, \lambda\}$ on the side of $\lambda^o$. Restrictions of a point set, Pareto optimal, in Figures 9B and 9C it is lowered to -0.5 (that restrictions are visible). This information is also a basis for further research of structure of a Pareto set (a set of options of structures of material).

3.1.5.5 Stage 5: Decision-making in the Structure of Material Model at the Set Priority of Criterion.

Algorithm 2. The solution of a vector task with a criterion priority.

The person making decisions, as a rule, is the designer of material.

Step 1. The solution of a VPMP with equivalent criteria. Results of the decision are presented in section 3.3. If results of the solution of a VPMP with equivalent criteria do not satisfy the person making the decision, then the choice of an optimal solution is carried out from any subset of points of $S^o_1, S^o_2, S^o_3, S^o_4$.

Step 2. Choice of priority criterion of $q \in K$. From the theory (the Theorem 2) it is known that in an optimum point of $X^o$ there are always two most contradictory criteria: $q \in K$ and $v \in K$ for which in the relative units precise equality is carried out: $\lambda^o = \lambda_q(X^o) = \lambda_p(X^o), q, p \in K, X \in S$, and for the others it is carried out inequalities: $\lambda^o \leq \lambda_k(X^o), \forall k \in K, q \neq p \neq k$.

For the choice of priority criterion on the display the message about results of the solution of $\lambda$-problem in physical and relative units is given: Criteria (6.88) in $X^o$ optimum point:

$$F(X^o) = \{f_1(X^o) = 346.5, f_2(X^o) = 26930, f_3(X^o) = 1342, f_4(X^o) = 50.2\}$$ (155)

The relative estimates (139) in $X^o$:

$$\lambda(X^o) = \{\lambda_1(X^o) = 0.546, \lambda_2(X^o) = 0.7235, \lambda_3(X^o) = 0.546, \lambda_4(X^o) = 0.546\}$$ (156)

From here it is visible (conclusion), in model of material (122)-(127) and the corresponding $\lambda$-problem (131)-(133) such criteria are the first, third and fourth:

$$\lambda^o = \lambda_1(X^o), \lambda_3(X^o), \lambda_4(X^o) = 0.546$$ (157)

Let's show the first $\lambda_1(X)$ and third $\lambda_3(X)$ criterion in Figure 9C on the basis of which we will conduct their further research.

From a pair of conflicting criteria, a criterion chosen by the decision maker would be improved. Such a criterion is called "priority criterion", we will designate it $q = 3 \in K$. This criterion is investigated in interaction with the first criterion of $q = 1 \in K$. On the display the message is given:

$q =$input ('Enter priority criterion (number) of $q =$') - Entered: $q = 3$. 

https://doi.org/10.53964/mit.2023007
Step 3. Numerical limits of change of size of a priority of criterion of $q=3 \in K$ are defined. For priority criterion of $q = 3 \in K$ changes of numerical limits in physical units upon transition from $X^p$ optimum point to the point of $X^p_q$ received on the first step at equivalent criteria are defined. $q=3$ given about criterion are given for the screen:

$$f_q(X^p) = 134.183 \leq f_q(X) \leq 150.238 = f_q(X^*_q), q \in K \quad (158)$$

In the relative units the criterion of $q=3$ changes in the following limits:

$$\lambda_q(X^p) = 0.546 \leq \lambda_q(X) \leq 1 = \lambda_q(X^*_q), q = 3 \in K \quad (159)$$

These data are analyzed.

Step 4. Choice of size of priority criterion of $q \in K$ in the VPMP. (Decision-making).

On the message: "Enter the size of priority criterion $fq$" - we enter the size of the characteristic defining structure of material: $f_q = 138.2$.

Step 5. The relative assessment is calculated.

For the chosen size of priority criterion $f_q = 138.2$ the relative assessment is calculated:

$$\lambda_q = \frac{\lambda_q(X^p) - \lambda_q(X)}{\lambda_q(X^p) - \lambda_q(X^*_q)} = \frac{0.546 - 0.8589}{0.8589 - 0.5534} = 0.6596 \quad (160)$$

which upon transition from $X^p$ point to $X^*_q$ lies in limits:

$$\lambda_q(X^p) = 0.546 \leq \lambda_q(X) \leq 0.6596 \leq \lambda_q(X^*_q), q = 3 \in K$$

Step 6. Let’s calculate coefficient of the linear approximation

Assuming the linear nature of change of criterion of $f_q(X)$ in (158) and according to the relative assessment of $\lambda_q(X)$, using reference methods of the linear approximation, we will calculate a constant of proportionality between $\lambda_q(X^p)$, $\lambda_q$ which we will call $\rho$:

$$\rho = \frac{\lambda_q(X^p) - \lambda_q(X)}{\lambda_q(X^p) - \lambda_q(X^*_q)} = \frac{0.8589 - 0.5534}{0.8589 - 0.5534} = 0.2502, q = 3 \in K \quad (161)$$

Step 7. Let’s calculate coordinates of a priority of criteria with dimension of $f_q$.

Assuming the linear nature of change of a vector of $X^q = \{x_1, x_3\}$, $q=3$ we will determine point coordinates with dimension of $f_q = 138.2$, the relative assessment (160):

$$X^q_{i=6.6596} = \{x_1 = X^p(1) + \rho(X^*_q(1) - X^p(1)), x_2 = X^p(2) + \rho(X^*_q(2) - X^p(2)), x_3 = X^p(3) + \rho(X^*_q(3) - X^p(3)), x_4 = X^p(4) + \rho(X^*_q(4) - X^p(4))\} \quad (162)$$

Where

$$X^p = \{x_1 = 69.5, x_2 = 24.1, x_3 = 4.144, x_4 = 2.2\}$$

$$X^*_3 = \{x_1 = 79, x_2 = 11.9, x_3 = 2.1, x_4 = 7\}$$

As result of the decision (6.90) we will receive $X^q$ point with coordinates:

$$X^q = \{x_1 = 71.877, x_2 = 21.05, x_3 = 3.673, x_4 = 3.4\}$$

Step 8. Calculation of the main indexes of a point of $X^q$.

For the received $X^q$ point, we will calculate:

all criteria in physical units
\[ f_k(X) = \{ f_k(X), k = 1, \ldots, K \} \]
\[ f(X) = \{ f_1(X) = 329.5, f_2(X) = 2841.8, f_3(X) = 137.4, f_4(X) = 46.6 \} \] (163)

All relative estimates of criteria,
\[ \lambda_k = \{ \lambda_k, k = 1, \ldots, K \} \]
\[ \lambda_k(X) = \frac{f_k(X) - f_k(X_0)}{f_k(X_0)}, k = 1, \ldots, K. \]
\[ \lambda_k(X) = \{ \lambda_1(X) = 0.1929, \lambda_2(X) = 0.5328, \lambda_3(X) = 0.7242, \lambda_4(X) = 0.7117 \} \] (164)

Step 9. Symmetry in the results of the solution of the structure of material represented by the vector problem of nonlinear programming.

In the considered vector problem of nonlinear programming (146)-(150) with four heterogeneous criteria - the model of the structure of material and the corresponding numerical variant (146) - (150) and built on its basis \( \lambda \)-problems we obtained the optimum point \( X^o \) and the maximum relative estimate \( \lambda^o \):
\[ X^o = \{ X^o, \lambda^o \} = \{ X^o = \{ x_1 = 69.5, x_2 = 24.1, x_3 = 4.144, x_4 = 2.2, \lambda^o = 0.546 \} \]

At the optimum point \( X^o \) the criteria in natural units are:
\[ F(X^o) = \{ f_1(X^o) = 346.5, f_2(X^o) = 2693.0, f_3(X^o) = 134.2, f_4(X^o) = 50.2 \} \]

and relative units:
\[ \lambda(X^o) = \{ \lambda_1(X^o) = 0.5461, \lambda_2(X^o) = 0.72, \lambda_3(X^o) = 0.5461, \lambda_4(X^o) = 0.5461 \} \]

This result confirms the proofs of theorem 1 about the most contradictory criteria in the vector optimization problem, i.e.,
\[ \lambda^o = \lambda_1(X^o) = \lambda_3(X^o) = \lambda_4(X^o) = 0.5461, \{ 1, 3, 4 \} \in K, X^o \in S \]

The rest of the criteria in relative units lie within:
\[ \{ \lambda_2(X^o) = 0.72 \} > \lambda^o \]

Theoretically, the point \( X^o \) is the center of symmetry. Indeed, the \( X^o \) point lies between the points of the optimum \( X_1^* \) and \( X_3^* \) obtained for each criterion (step 1), in which \( \lambda_1(X_1^*) = 1, \lambda_3(X_3^*) = 1 \). Let's show geometrically symmetry in Figure 9D.

For the three criteria first \( \lambda_1(X) \) , third \( \lambda_3(X) \) , and fourth \( \lambda_4(X) \) , " this example considers odd numerical symmetry.

3.1.5.6 Stage 6: Geometrical Interpretation of Results of the Decision in the Structure of Material in Three to a Measured Frame in Physical Units

The solution of a problem of non-linear programming (146), (150) in three-dimensional frames of \( x_1, x_3 \) and \( f_1(X) \) is presented on Figure 10A. \( f_2(X) \) is presented on Figure 10B.
Figure 9. Result the solution of the problem (122), (126)-(127) in the three-dimensional coordinate system. A: \{x_1, x_3 and f_1(X)\}; B: \{x_1, x_3 and f_2(X)\}; C: \{x_1, x_3 and f_3(X)\}; D: \{x_1, x_3 and f_1(X)\}.

\[ X_3 = \{x_1 = 79, x_2 = 11.9, x_3 = 2.1, x_4 = 7\}, f_3^* = f_3(X_3) = 150.24 \]

The solution of a problem of non-linear programming (123), (126)-(127) in three-dimensional frames of \(x_1, x_3\) and \(f_3(X)\) is presented on Figure 10C.

\[ X_4 = \{x_1 = 36.7, x_2 = 59, x_3 = 2.1, x_4 = 2.2\}, f_4^* = f_4(X_4) = 30.71 \]

The solution of a problem of non-linear programming (149), (150) in three-dimensional frames of \(x_1, x_3\) and \(f_4(X)\) is presented on Figure 10D.

Analysis of results in the VPMP. The calculated size of criterion \(f_q(X^*_q)\), \(q \in K\) is usually not equal to the set \(f_q\). The error of the choice of \(\Delta f_q = |f_q(X^*_q) - f_q| = |137.4 - 138.2| = 0.8\) is defined by an error of linear approximation, \(\Delta f_q\% = 0.04\%\).

In the course of modeling parametric restrictions (150) can be changed, i.e., some set of optimum decisions is received. Choose a final version which in our example included from this set of optimum decisions:

parameters of material:

\[ X^0 = \{x_1 = 69.5, x_2 = 24.1, x_3 = 4.144, x_4 = 2.2\} \]

\[ X^{\text{summa}} = x_1 + x_2 + x_3 + x_4 = 100 \]

the parameters of the material at a given priority criterion \(q=3:\)

\[ X^q = \{x_1 = 71.877, x_2 = 21.05, x_3 = 3.673, x_4 = 3.4\} \]

\[ X^{\text{summa}} = x_1 + x_2 + x_3 + x_4 = 100 \]
4 CONCLUSION

The study of the processes of digital transformation of optimal decision-making in economic and engineering systems based on the theory and methods of vector optimization is one of the most important tasks of system analysis and innovative design. The main contributions of this paper can be summarized as follows: first, the digital transformation of the mathematical model of the engineering system in conditions of certainty and uncertainty in the form of a vector problem of mathematical programming; Secondly, on the basis of the theory, constructive methods for solving vector optimization problems have been developed, which make it possible to make a decision with equivalent criteria and with a given priority of the criterion. The practice of "making an optimal decision" on the basis of a mathematical model is presented on the example of solving vector problems of mathematical (linear) programming, which are used in economic problems of modeling (forecasting, planning) of the development of the firm, market, region. Constructive methods for solving vector problems of mathematical (convex) programming, using experimental data, can be used in the design of technical systems, technological processes and material structure in various industries. The author is ready to participate in solving vector problems of linear and nonlinear programming.

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Conflicts of Interest
The author declared no conflict of interest.

Author Contribution
The author contributed to writing the article, read and approved its submission.

Abbreviation List
VPLP, Vector problem of linear programming
VPMP, Vector problem of mathematical programming

References


