 https://doi.org/10.53964/mem.2024014

1 DYNAMIC PROBLEM OF THE MIDDLEMAN/FINANCIAL INTERMEDIARY

This part addresses the dynamic problem of financial intermediaries. They aim to maximize the present discounted value of an infinite series of future profits in such an environment. The present value at time zero is outlined as follows:

$$V(0) = \int_0^{\infty} [rq(r) - \varsigma(m(r,e))]e^{-\sigma t}dt \ (53)$$

The discount rate, control variable, and state variable are represented by and respectively. The maximization problem is expressed in mathematical notation as shown below:

$$\underset{\{r(t)\}}{Max}V(0) = \int_0^{\infty} [rq(r) - \varsigma(m(r,e))]e^{-\sigma t}dt,$$

while satisfying the following conditions:

$\dot{m}(t) = m_1^{'}(r(t), e(r(t), z))\dot{r}(t) + m_2^{'}(r(t), e(r(t), z))e_1^{'}(r(t), z)\dot{r}(t)$ (state equation illustrates the evolution of the state variable over time; are exogenous factors),

$m(0) = m_s$ (initial condition),

$m(t) \geq 0$ (the requirement that the state variable must be non-negative),

$m(\infty)$ free (terminal condition).

Below is the current-value Hamiltonian expression:

$$\widetilde{H} = r(t)q(r(t)) - \varsigma(m(r(t), e(r(t), z))) + \mu(t)\dot{r}(t)\begin{bmatrix} m_1^{'}(r(t), e(r(t), z)) + m_2^{'}(r(t), e(r(t), z)) * \\ e_1^{'}(r(t), z) \end{bmatrix} \ (54)$$

The conditions for maximization are provided as follows:

(i) maximizes for all t: $\frac{\partial \widetilde{H}}{\partial r} = 0$,

(ii) $\dot{\mu} - \sigma\mu = -\frac{\partial \widetilde{H}}{\partial m}$,

(iii) (this simply returns the state equation),

(iv) (the transversality condition).

The first and second conditions are outlined below:

$$\frac{\partial \widetilde{H}}{\partial r} = q(r(t)) + r(t)q^{'}(r(t)) - \varsigma^{'}(m(r(t), e(r(t), z)))\begin{Bmatrix} m_1^{'}(r(t), e(r(t), z)) + m_2^{'}(r(t), e(r(t), z)) * \\ e_1^{'}(r(t), z) \end{Bmatrix}$$
$$+\mu(t)\dot{r}(t) * \begin{bmatrix} m_{11}^{''}(r(t), e(r(t), z)) + m_{12}^{''}(r(t), e(r(t), z))e_1^{'}(r(t), z) + \\ m_{21}^{''}(r(t), e(r(t), z))e_1^{'}(r(t), z) + m_{22}^{''}(r(t), e(r(t), z))e_1^{'2}(r(t), z) + \\ m_2^{'}(r(t), e(r(t), z))e_{11}^{''}(r(t), z) \end{bmatrix} = 0 \ (55)$$

and

$$\dot{\mu} - \sigma\mu = -\frac{\partial \widetilde{H}}{\partial m} = \varsigma^{'}(m(r(t), e(r(t), z))) \ (56)$$

In equilibrium, and expression boils down to the following:

$$q(r(t)) + r(t)q^{'}(r(t)) - \varsigma^{'}(m(r(t), e(r(t), z)))\begin{Bmatrix} m_1^{'}(r(t), e(r(t), z)) + m_2^{'}(r(t), e(r(t), z)) * \\ e_1^{'}(r(t), z) \end{Bmatrix}$$

$$= 0,$$

$$r(t)q^{'}(r(t)) + q(r(t)) = \varsigma^{'}(m(r(t), e(r(t), z))) \begin{Bmatrix} m_{1}^{'}(r(t), e(r(t), z)) + m_{2}^{'}(r(t), e(r(t), z)) * \\ e_{1}^{'}(r(t), z) \end{Bmatrix},$$

$$r(t)\left[1 + \frac{1}{demand\ elasticity}\right] = \varsigma^{'}(m(r(t), e(r(t), z))) \left\{\frac{m_{1}^{'}(r(t), e(r(t), z))}{q^{'}(r(t))} + \frac{m_{2}^{'}(r(t), e(r(t), z))e_{1}^{'}(r(t), z)}{q^{'}(r(t))}\right\}$$

This suggests that when demand is infinitely elastic, the interest rate equals the marginal cost. However, the marginal cost, depicted on the right-hand side of the equation, differs from that in a myopic problem because in dynamic considerations, the financial intermediary also considers the impact of the market interest rate on the purchase interest rate charged by producers. In the event of a positive supply shock, the marginal cost of holding an additional unit of funds in inventory increases for the financial intermediary, as the term is higher at that particular time with the existing interest rate. The term remains unchanged as the interest rate remains the same as before. At the current interest rate, the financial intermediary encounters the following inequality:

$$\frac{\partial \widetilde{H}}{\partial r} = q(r(t)) + r(t)q^{'}(r(t)) - \varsigma^{'}(m(r(t), e(r(t), z))) \begin{Bmatrix} m_{1}^{'}(r(t), e(r(t), z)) + m_{2}^{'}(r(t), e(r(t), z)) * \\ e_{1}^{'}(r(t), z) \end{Bmatrix}$$

$$+\mu(t)\dot{r}(t) * \begin{bmatrix} m_{11}^{''}(r(t), e(r(t), z)) + m_{12}^{''}(r(t), e(r(t), z))e_{1}^{'}(r(t), z) + \\ m_{21}^{''}(r(t), e(r(t), z))e_{1}^{'}(r(t), z) + m_{22}^{''}(r(t), e(r(t), z))e_{1}^{'2}(r(t), z) + \\ m_{2}^{'}(r(t), e(r(t), z))e_{11}^{''}(r(t), z) \end{bmatrix} < 0.$$

To fulfill the dynamic optimization condition, the financial intermediary needs to decrease the interest rate in order to include another unit of funds in its inventory. This indicates a negative correlation between the inventory of funds and the interest rate. The concepts of demand and supply are linked through inventory; when their rates match, market equilibrium is achieved. However, if there's a disparity between their rates and other agents do not respond to this change, the financial intermediary will continuously adjust the interest rate until market saturation occurs. This market behavior can be articulated as follows: $\text{Change in interest rate} \propto \text{change in market (funds)inventory}. R = \text{change in interest rate}.$

$$M = m - m_s = \text{change in market inventory of funds},$$

$$m = \text{funds inventory at time } t,$$

$$m_s = \text{funds inventory in steady state equilibrium}.$$

$$\text{Input} - \text{output} = \frac{dm}{dt} = \frac{d(m - m_s)}{dt} = \frac{dM}{dt},$$

$$\text{or } M = \int (\text{input} - \text{output})dt.$$

$$\text{Change in interest rate} \propto \int (\text{supply rate} - \text{demand rate})dt, \text{or}$$

$$R = -K_m \int (\text{supply rate} - \text{demand rate})dt,$$

The constant represents proportionality. The negative sign indicates that when the supply rate exceeds the demand rate, becomes negative, i.e., a decrease in the interest rate. This expression can also be represented as:

$$\int (\text{supply rate} - \text{demand rate})dt = -\frac{R}{K_m}, \text{or} \int (w_i - w_0)dt = -\frac{R}{K_m} \ (57)$$

w_i = supply rate,

$w_0 = demand\ rate,$

K_m = dimensional constant.

At $t = 0$, when the *supply rate* equals the *demand rate*, indicating market equilibrium, Equation can be written as:

$$\int (w_{is} - w_{0s})dt = 0 \ (58)$$

The subscript represents a value in a steady-state equilibrium, where $R = 0$. By subtracting equation from Equation (57), we get:

$$\int (w_i - w_{is})dt - \int (w_0 - w_{0s})dt = -\frac{R}{K_m}, \text{or} \int (W_i - W_0)dt = -\frac{R}{K_m} \ (59)$$

where $w_i - w_{is} = W_i$ = change in supply rate,

$w_0 - w_{0s} = W_0$ = change in demand rate.

R, and are deviation variables, indicating deviations from the steady-state equilibrium with initial values of zero. Equation can also be represented as:

$$R = -K_m \int W dt = -K_m M \ (60)$$

where If experiences a sudden change due to a factor unrelated to a change in fund inventory, this additional input can be incorporated into equation as follows:

$$R = -K_m \int W dt + B = -K_m M + B \ (59a)$$

In addition to the feedback effect of the interest rate, the funds inventory can also be subjected to an exogenous shock.

2 SOLUTION OF THE MODEL WITH AN EXPANSIONARY MONETARY POLICY

Expressions from Equation (11a), and respectively are stated below:

$$\frac{dR(t)}{dt} = -K_m W(t),$$

$$W_d(t) = -K_d R(t),$$

$$W_{mp} = -K_{sp}(C_p - R),$$

$$W_{mc} = -K_{sc}(C_c - R).$$

and

$$W(t) = W_m(t) - W_d(t),$$

if no exogenous demand or supply shock happens. The total money supply encompasses the supply from various sources including the central bank, households, and firms. It can be delineated as a combination of two primary sources: the central bank and the public (which includes households, firms, etc.), represented as follows:

$$W_m(t) =- K_{sp}\left[C_p(t) - R(t)\right] - K_{sc}[C_c(t) - R(t)] \text{ (61)}$$

Subscripts and have been incorporated to distinguish between the public and the central bank respectively. We can merge the aforementioned expressions to form:

$$\frac{dR(t)}{dt} =- K_m[W_m(t) - W_d(t)]$$

$$=- K_m\left[- K_{sp}\{C_p(t) - R(t)\} - K_{sc}\{C_c(t) - R(t)\} + K_d R(t)\right]$$

$$=- K_m\left[- K_{sp}C_p(t) - K_{sc}C_c(t) + (K_{sp} + K_{sc} + K_d)R(t)\right]$$

Rearranging above expression gives:

$$\frac{dR(t)}{dt} + K_m(K_{sp} + K_{sc} + K_d)R(t) = K_m\left[K_{sp}C_p(t) + K_{sc}C_c(t)\right] \text{ (62)}$$

In the money market, unlike the goods market, the central bank assumes the dual roles of the government and a fund producer. When the central bank lowers the interest rate on discount loans, its supply curve shifts downward, leading to a cost in its role as the government. To maintain clarity regarding the central bank's actions in the money market, it's crucial to distinguish between its roles as a fund producer and as a government entity. Let's denote the change in cost incurred by the central bank due to the reduction in interest rates on discount loans as A, while assuming the cost of the public's money supply remains constant. The above equation can thus be represented as:

$$\frac{dR(t)}{dt} + K_m(K_{sp} + K_{sc} + K_d)R(t) =- K_m K_{sc} A \text{ (63)}$$

The solution is given by the following expression:

$$R(t) =- \frac{K_{sc}A}{(K_{sp}+K_{sc}+K_d)} + \frac{K_{sc}A}{(K_{sp}+K_{sc}+K_d)} e^{-[K_m(K_{sp}+K_{sc}+K_d)]t} \text{ (64)}$$

$R(0) = 0$ (initial condition), and (final steady state equilibrium value). As a result of a monetary policy, the interest rate dynamics depends on parameters and A.

3 A DYNAMIC OPTIMAL EXPANSIONARY MONETARY POLICY

Depending on whether the monetary policy leans towards expansion or contraction, the post-policy equilibrium may see either efficiency gains or losses compared to the initial state. However, these aren't the sole determinants of efficiency shifts; adjustments in the money market during the transition to the final equilibrium also contribute to efficiency changes. When the monetary policy is enacted, the money supply either increases or decreases while demand remains constant at the initial interest rate, disrupting the money market's equilibrium. Subsequently, the adjustment of interest rates commences to reconcile the supply and demand and establish a new equilibrium in the money market. The post-policy equilibrium interest rate is influenced by the elasticity of demand and supply. As previously discussed, the alteration in total supply resulting from monetary policy is as follows:

$$W_m(0) =- K_{sp}\left[C_p(0) - R(0)\right] - K_{sc}[C_c(0) - R(0)] = K_{sc}A \text{ (65)}$$

as $R(0) = 0$

Due to monetary policy, the money supply either increases or decreases by $K_{sc}A$. Since demand remains constant, the money inventory also changes by $K_{sc}A$, aligning with the change in supply. Consequently, the market falls out of equilibrium, prompting market forces to drive the money market towards its final equilibrium by adjusting interest rates. As interest rate fluctuates, the demand and supply of money respond in kind through feedback mechanisms. A surplus in the money inventory indicates an excess supply compared to demand, and vice versa. Efficiency loss occurs when the money market is out of equilibrium, signifying that either the supply or consumption of money/funds is inefficient at that moment. This implies that the total efficiency loss during the adjustment of the money market is the cumulative difference between supply and demand at all points in time. After factoring in any efficiency loss in the post-policy equilibrium compared to the initial/pre-policy equilibrium, the total efficiency loss for expansionary and contractionary monetary policies respectively can be expressed as:

$$EL(\text{expansionary}) = \int_{-\infty}^{0} W_m(\infty)dt + \int_{0}^{\infty} [W_m(t) - W_d(t)]dt$$

$$= \int_{-\infty}^{0} W_m(\infty)dt + M(t) \ (66)$$

$$EL(\text{contractionary}) = \left| \int_{t}^{\infty} W_m(\infty)dt + \int_{0}^{t} W_d(t)dt + M(t) \right|$$

$$= \left| \int_{0}^{t} W_d(t)dt + \int_{t}^{\infty} W_d(\infty)dt + M(t) \right|$$

$$= \left| \int_{0}^{\infty} W_d(t)dt + \int_{0}^{\infty} [W_m(t) - W_d(t)]dt \right|$$

$$= \left| \int_{0}^{\infty} W_m(t)dt \right| \ (67)$$

Eq. states the following:

$$R(t) =- K_m M(t) + B$$

By imposing the initial conditions, we can determine the value of (for expansionary monetary policy) as follows:

$$R(0) =- K_m M(0) + B,$$

$$0 =- K_m K_{sc} A + B,$$

$$B = K_m K_{sc} A.$$

After plugging in the above expression in Equation it transforms to

$$R(t) =- K_m M(t) + K_m K_{sc} A, \text{or}$$

$$M(t) =- \frac{1}{K_m}[R(t) - K_m K_{sc} A]$$

With Expansionary Monetary Policy Cost Constraint:

Referring to Equation the change in money supply resulting from a shift in the market interest rate is as follows:

$$W_m(t) =- K_{sp}\left[C_p(t) - R(t)\right] - K_{sc}[C_c(t) - R(t)]$$

The part of the supply coming from the central bank is and may be written as follows:

$$W_{mc}(t) =- K_{sc}[C_c(t) - R(t)],$$
$$w_{nmc}(t) - w_{imc}(0) =- K_{sc}[C_c(t) - R(t)],$$

$w_{imc}(0)$ represents the initial money supply provided by the central bank, while denotes the adjusted money supply after the central bank implements its monetary policy. The expression indicates the deviation from the initial equilibrium value over time. The cost of the monetary policy can be articulated as:

$$MPC = A[w_{imc}(0) + K_{sc}\{A + R(t)\}] \text{ (68)}$$

The challenge of minimizing efficiency loss while adhering to a constraint on monetary policy cost can be depicted as follows:

$$\min_A EL \text{ s.t. } MPC \le G$$

G represents the central bank's cost associated with implementing monetary policy. The variable to be chosen is the monetary policy itself, denoted as A, and the constraint becomes effective at time $t = 0$. The Lagrangian for the aforementioned problem is presented below:

$$\mathcal{L} = \int_{-\infty}^{0} W_m(\infty)dt + M(t) + \lambda\big[G - A[w_{imc}(0) + K_{sc}\{A + R(t)\}]\big]$$

$$= \int_{-\infty}^{0} \left[K_{sc}A - \frac{K_{sc}(K_{sp} + K_{sc})A}{(K_{sp} + K_{sc} + K_d)}\right] dt$$

$$-\frac{1}{K_m}\left[-\frac{K_{sc}A}{(K_{sp} + K_{sc} + K_d)} + \frac{K_{sc}A}{(K_{sp} + K_{sc} + K_d)} e^{-[K_m(K_{sp}+K_{sc}+K_d)]t} - K_m K_{sc} A\right]$$

$$+\lambda\left[G - A\left[w_{imc}(0) + K_{sc}\left\{A - \frac{K_{sc}A}{(K_{sp} + K_{sc} + K_d)} + \frac{K_{sc}A}{(K_{sp} + K_{sc} + K_d)} e^{-[K_m(K_{sp}+K_{sc}+K_d)]t}\right\}\right]\right]$$

$$= \int_{-\infty}^{0} \frac{K_{sc}K_d A}{(K_{sp} + K_{sc} + K_d)} dt - \frac{1}{K_m}\left[-\frac{K_{sc}A}{(K_{sp} + K_{sc} + K_d)} + \frac{K_{sc}A}{(K_{sp} + K_{sc} + K_d)} e^{-[K_m(K_{sp}+K_{sc}+K_d)]t} - K_m K_{sc} A\right]$$

$$+\lambda\left[G - A\left[w_{imc}(0) + K_{sc}\left\{A - \frac{K_{sc}A}{(K_{sp} + K_{sc} + K_d)} + \frac{K_{sc}A}{(K_{sp} + K_{sc} + K_d)} e^{-[K_m(K_{sp}+K_{sc}+K_d)]t}\right\}\right]\right]$$

Differentiating the Lagrangian with respect to yields the following first-order derivative:

$$\int_{-\infty}^{0} \frac{K_{sc}K_d}{(K_{sp} + K_{sc} + K_d)} dt - \frac{1}{K_m}\left[-\frac{K_{sc}}{(K_{sp} + K_{sc} + K_d)} + \frac{K_{sc}}{(K_{sp} + K_{sc} + K_d)} e^{-[K_m(K_{sp}+K_{sc}+K_d)]t} - K_m K_{sc}\right]$$

$$-\lambda\left[w_{imc}(0) + K_{sc}\left\{A - \frac{K_{sc}A}{(K_{sp} + K_{sc} + K_d)} + \frac{K_{sc}A}{(K_{sp} + K_{sc} + K_d)} e^{-[K_m(K_{sp}+K_{sc}+K_d)]t}\right\}\right]$$

$$-\lambda A K_{sc}\left[1 - \frac{K_{sc}}{(K_{sp} + K_{sc} + K_d)} + \frac{K_{sc}}{(K_{sp} + K_{sc} + K_d)} e^{-[K_m(K_{sp}+K_{sc}+K_d)]t}\right] = 0$$

Rearranging this, we get:

$$\int_{-\infty}^{0} \frac{K_{sc}K_d}{(K_{sp}+K_{sc}+K_d)}dt - \frac{1}{K_m}\left[-\frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)} + \frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)}e^{-[K_m(K_{sp}+K_{sc}+K_d)]t} - K_m K_{sc}\right]$$

$$-2\lambda A K_{sc}\left[1 - \frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)} + \frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)}e^{-[K_m(K_{sp}+K_{sc}+K_d)]t}\right] = \lambda w_{imc}(0),$$

or

$$A =- \frac{\lambda w_{imc}(0) - \left[\int_{-\infty}^{0} \frac{K_{sc}K_d}{(K_{sp}+K_{sc}+K_d)}dt - \frac{1}{K_m}\left[-\frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)} + \frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)}e^{-[K_m(K_{sp}+K_{sc}+K_d)]t} - K_m K_{sc}\right]\right]}{2\lambda K_{sc}\left[1 - \frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)} + \frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)}e^{-[K_m(K_{sp}+K_{sc}+K_d)]t}\right]} \text{ (69)}$$

The first-order derivative with respect to is as follows:

$$G - A\left[w_{imc}(0) + K_{sc}\left\{A - \frac{K_{sc}A}{(K_{sp}+K_{sc}+K_d)} + \frac{K_{sc}A}{(K_{sp}+K_{sc}+K_d)}e^{-[K_m(K_{sp}+K_{sc}+K_d)]t}\right\}\right] = 0 \text{ (70)}$$

Putting Equation into Equation (70), we get:

$$G =$$

$$- w_{imc}(0). \frac{\lambda w_{imc}(0) - \left[\int_{-\infty}^{0} \frac{K_{sc}K_d}{(K_{sp}+K_{sc}+K_d)}dt - \frac{1}{K_m}\left[-\frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)} + \frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)}e^{-[K_m(K_{sp}+K_{sc}+K_d)]t} - K_m K_{sc}\right]\right]}{2\lambda K_{sc}\left[1 - \frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)} + \frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)}e^{-[K_m(K_{sp}+K_{sc}+K_d)]t}\right]}$$

$$+ K_{sc}\left\{1 - \frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)} + \frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)}e^{-[K_m(K_{sp}+K_{sc}+K_d)]t}\right\}$$

$$* \left[- \frac{\lambda w_{imc}(0) - \left[\int_{-\infty}^{0} \frac{K_{sc}K_d}{(K_{sp}+K_{sc}+K_d)}dt - \frac{1}{K_m}\left[-\frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)} + \frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)}e^{-[K_m(K_{sp}+K_{sc}+K_d)]t} - K_m K_{sc}\right]\right]}{2\lambda K_{sc}\left[1 - \frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)} + \frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)}e^{-[K_m(K_{sp}+K_{sc}+K_d)]t}\right]}\right]^2,$$

$$\text{or} 4\lambda^2 QG =- 2\lambda^2 w_{imc}^2(0) + 2\lambda w_{imc}(0)J + \lambda^2 w_{imc}^2(0) + J^2 - 2\lambda w_{imc}(0)J,$$

$$\text{where} Q = K_{sc}\left[1 - \frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)} + \frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)}e^{-[K_m(K_{sp}+K_{sc}+K_d)]t}\right],$$

$$J = \int_{-\infty}^{0} \frac{K_{sc}K_d}{(K_{sp}+K_{sc}+K_d)}dt - \frac{1}{K_m}\left[-\frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)} + \frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)}e^{-[K_m(K_{sp}+K_{sc}+K_d)]t} - K_m K_{sc}\right]$$

or

$$\{w_{imc}^2(0) + 4QG\}\lambda^2 - J^2 = 0$$

$$\lambda = \frac{J}{\sqrt{w_{imc}^2(0) + 4QG}}$$

λ is a positive quantity since an increase in leads to a rise in the minimum efficiency loss. Referencing Equation (69):

$$A =- \frac{\lambda w_{imc}(0) - J}{2\lambda Q} \text{ (71)}$$

By replacing with its value in the above expression, we get:

$$A =- \frac{\frac{w_{imc}(0)J}{\sqrt{w_{imc}^2(0)+4QG}} - J}{\frac{2QJ}{\sqrt{w_{imc}^2(0)+4QG}}}, A =- \frac{w_{imc}(0) - \sqrt{w_{imc}^2(0)+4QG}}{2Q} \text{ (72)}$$

The second-order condition for confirming minimization can be verified as follows:

$$\mathcal{L} = JA + \lambda[G - A(w_{ime}(0) + QA)]$$

The bordered Hessian matrix of the Lagrange function is presented below:

$$BH = \begin{bmatrix} 0 & w_{imc}(0) + 2QA \\ w_{imc}(0) + 2QA & \dfrac{-2QJ}{\sqrt{w_{imc}^2(0) + 4QG}} \end{bmatrix}$$

The determinant of the matrix is negative because indicating that efficiency loss has been minimized.

4 SOLUTION OF THE MODEL WITH A CONTRACTIONARY MONETARY POLICY

The expressions from Equation (11a), and are as follows:

$$\frac{dR(t)}{dt} =- K_m W(t),$$

$$W_d(t) =- K_d R(t),$$

$$W_{mp} =- K_{sp}(C_p - R),$$

$$W_{mc} =- K_{sc}(C_c - R).$$

and

$$W(t) = W_m(t) - W_d(t),$$

if no exogenous demand or supply shock happens. The variable represents the total money supply, including contributions from the central bank, households, firms, etc. It can be expressed as a combination of two sources of money supply: the central bank and the public (which includes households, firms, etc.), as illustrated below:

$$W_m(t) =- K_{sp}[C_p(t) - R(t)] - K_{sc}[C_c(t) - R(t)] \quad (73)$$

The subscripts and represent the public and the central bank respectively. By consolidating the preceding expressions, we can formulate:

$$\frac{dR(t)}{dt} =- K_m[W_m(t) - W_d(t)]$$

$$=- K_m[- K_{sp}\{C_p(t) - R(t)\} - K_{sc}\{C_c(t) - R(t)\} + K_d R(t)]$$

$$=- K_m[- K_{sp}C_p(t) - K_{sc}C_c(t) + (K_{sp} + K_{sc} + K_d)R(t)]$$

Rearranging the above expression gives:

$$\frac{dR(t)}{dt} + K_m(K_{sp} + K_{sc} + K_d)R(t) = K_m[K_{sp}C_p(t) + K_{sc}C_c(t)] \quad (74)$$

When the central bank raises the interest rate on discount loans, its supply curve shifts to the left, generating revenue for the government. Let's denote the change in the central bank's costs resulting from the increased interest rate on discount loans as while assuming that the cost of the public's money supply remains unchanged. This equation can thus be expressed as follows:

$$\frac{dR(t)}{dt} + K_m(K_{sp} + K_{sc} + K_d)R(t) = K_m K_{sc} A \ (75)$$

The solution is given by the following expression:

$$R(t) = \frac{K_{sc}A}{(K_{sp}+K_{sc}+K_d)} - \frac{K_{sc}A}{(K_{sp}+K_{sc}+K_d)} e^{-[K_m(K_{sp}+K_{sc}+K_d)]t} \ (76)$$

$R(0) = 0$ (initial condition), and (final equilibrium value in a steady state).

5 A DYNAMIC OPTIMAL CONTRACTIONARY MONETARY POLICY

According to Equation (31), the alteration in the total supply stemming from a contractionary monetary policy can be expressed as:

$$W_m(0) =- K_{sp}\left[C_p(0) - R(0)\right] - K_{sc}[C_c(0) - R(0)] =- K_{sc}A, \text{as } R(0) = 0$$

Eq. states the following:

$$R(t) =- K_m M(t) + B$$

Applying the initial conditions allows us to ascertain the specific value of (for contractionary monetary policy) as follows:

$$R(0) =- K_m M(0) + B,$$
$$0 = K_m K_{sc} A + B,$$
$$B =- K_m K_{sc} A.$$

After substituting the aforementioned expression into Equation (11a), it undergoes transformation to:

$$R(t) =- K_m M(t) - K_m K_{sc} A, \text{or } M(t) =- \frac{1}{K_m}[R(t) + K_m K_{sc} A] \ (77)$$

With Contractionary Monetary Policy Revenue Constraint:

Referring to Equation the alteration in money supply resulting from a shift in the market interest rate is given by:

$$W_m(t) =- K_{sp}\left[C_p(t) - R(t)\right] - K_{sc}[C_c(t) - R(t)]$$

The part of the supply coming from the central bank is and can be written as follows:

$$W_{mc}(t) =- K_{sc}[C_c(t) - R(t)],$$
$$w_{nmc}(t) - w_{imc}(0) =- K_{sc}[C_c(t) - R(t)],$$

where represents the initial money supply from the central bank, and denotes the new money supply after the central bank implements the monetary policy. $W_{mc}(t) = w_{nmc}(t) - w_{imc}(0)$, indicating the deviation from the initial steady-state equilibrium value. The revenue gained by the government due to a contractionary monetary policy can be formulated as:

$$MPR = A[w_{imc}(0) - K_{sc}\{A - R(t)\}] \ (78)$$

The problem of minimizing efficiency loss while adhering to a monetary policy constraint is outlined as follows:

$$\min_{T} EL \text{ s.t. } MPR \geq G$$

G represents the income generated by the government through the implementation of monetary policy. The decision variable is the monetary policy denoted as A, and the constraint is binding at time $t = 0$. The Lagrangian expression for this problem can be formulated as follows:

$$\mathcal{L} = -\int_0^{\infty} [K_{sp}\{C_p(t) - R(t)\} + K_{sc}\{C_c(t) - R(t)\}]dt + \lambda[G - A[w_{imc}(0) - K_{sc}\{A - R(t)\}]]$$

$$= \int_0^{\infty} \left[- K_{sc}A + \frac{K_{sc}(K_{sp} + K_{sc})A}{(K_{sp} + K_{sc} + K_d)} - \frac{K_{sc}(K_{sp} + K_{sc})A}{(K_{sp} + K_{sc} + K_d)} e^{-[K_m(K_{sp}+K_{sc}+K_d)]t}\right] dt$$

$$+\lambda\left[G - A\left[w_{imc}(0) - K_{sc}\left\{A - \frac{K_{sc}A}{(K_{sp} + K_{sc} + K_d)} + \frac{K_{sc}A}{(K_{sp} + K_{sc} + K_d)} e^{-[K_m(K_{sp}+K_{sc}+K_d)]t}\right\}\right]\right]$$

The derivative of the Lagrangian with respect to yields the following expression:

$$\int_0^{\infty} \left[- K_{sc} + \frac{K_{sc}(K_{sp} + K_{sc})}{(K_{sp} + K_{sc} + K_d)} - \frac{K_{sc}(K_{sp} + K_{sc})}{(K_{sp} + K_{sc} + K_d)} e^{-[K_m(K_{sp}+K_{sc}+K_d)]t}\right] dt$$

$$-\lambda\left[w_{imc}(0) - K_{sc}\left\{A - \frac{K_{sc}A}{(K_{sp} + K_{sc} + K_d)} + \frac{K_{sc}A}{(K_{sp} + K_{sc} + K_d)} e^{-[K_m(K_{sp}+K_{sc}+K_d)]t}\right\}\right]$$

$$+\lambda AK_{sc}\left[1 - \frac{K_{sc}}{(K_{sp} + K_{sc} + K_d)} + \frac{K_{sc}}{(K_{sp} + K_{sc} + K_d)} e^{-[K_m(K_{sp}+K_{sc}+K_d)]t}\right] = 0$$

Rearranging this, we get:

$$\int_0^{\infty} \left[- K_{sc} + \frac{K_{sc}(K_{sp} + K_{sc})}{(K_{sp} + K_{sc} + K_d)} - \frac{K_{sc}(K_{sp} + K_{sc})}{(K_{sp} + K_{sc} + K_d)} e^{-[K_m(K_{sp}+K_{sc}+K_d)]t}\right] dt$$

$$+2\lambda AK_{sc}\left[1 - \frac{K_{sc}}{(K_{sp} + K_{sc} + K_d)} + \frac{K_{sc}}{(K_{sp} + K_{sc} + K_d)} e^{-[K_m(K_{sp}+K_{sc}+K_d)]t}\right] = \lambda w_{imc}(0),$$

or

$$A = \frac{\lambda w_{imc}(0) - \int_0^{\infty} \left[-K_{sc} + \frac{K_{sc}(K_{sp}+K_{sc})}{(K_{sp}+K_{sc}+K_d)} - \frac{K_{sc}(K_{sp}+K_{sc})}{(K_{sp}+K_{sc}+K_d)} e^{-[K_m(K_{sp}+K_{sc}+K_d)]t}\right] dt}{2\lambda K_{sc}\left[1 - \frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)} + \frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)} e^{-[K_m(K_{sp}+K_{sc}+K_d)]t}\right]} \quad (79)$$

The first-order derivative with respect to is provided below:

$$G - A\left[w_{imc}(0) - K_{sc}\left\{A - \frac{K_{sc}A}{(K_{sp}+K_{sc}+K_d)} + \frac{K_{sc}A}{(K_{sp}+K_{sc}+K_d)} e^{-[K_m(K_{sp}+K_{sc}+K_d)]t}\right\}\right] = 0 \quad (80)$$

Putting Equation into Equation (80), we obtain:

$$G = w_{imc}(0). \frac{\lambda w_{imc}(0) - \int_0^{\infty} \left[-K_{sc} + \frac{K_{sc}(K_{sp}+K_{sc})}{(K_{sp}+K_{sc}+K_d)} - \frac{K_{sc}(K_{sp}+K_{sc})}{(K_{sp}+K_{sc}+K_d)} e^{-[K_m(K_{sp}+K_{sc}+K_d)]t}\right] dt}{2\lambda K_{sc}\left[1 - \frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)} + \frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)} e^{-[K_m(K_{sp}+K_{sc}+K_d)]t}\right]}$$

$$- K_{sc}\left\{1 - \frac{K_{sc}}{(K_{sp} + K_{sc} + K_d)} + \frac{K_{sc}}{(K_{sp} + K_{sc} + K_d)} e^{-[K_m(K_{sp}+K_{sc}+K_d)]t}\right\}$$

$$*\left[\frac{\lambda w_{imc}(0)-\int_0^{\infty}\left[-K_{sc}+\frac{K_{sc}(K_{sp}+K_{sc})}{(K_{sp}+K_{sc}+K_d)}-\frac{K_{sc}(K_{sp}+K_{sc})}{(K_{sp}+K_{sc}+K_d)}e^{-[K_m(K_{sp}+K_{sc}+K_d)]t}\right]dt}{2\lambda K_{sc}\left[1-\frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)}+\frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)}e^{-[K_m(K_{sp}+K_{sc}+K_d)]t}\right]}\right]^2,$$

$$\text{or } 4\lambda^2 QG = 2\lambda^2 w_{imc}^2(0) - 2\lambda w_{imc}(0)J - \lambda^2 w_{imc}^2(0) - J^2 + 2\lambda w_{imc}(0)J,$$

$$\text{where } Q = K_{sc}\left[1-\frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)}+\frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)}e^{-[K_m(K_{sp}+K_{sc}+K_d)]t}\right],$$

$$J = \int_0^{\infty}\left[-K_{sc}+\frac{K_{sc}(K_{sp}+K_{sc})}{(K_{sp}+K_{sc}+K_d)}-\frac{K_{sc}(K_{sp}+K_{sc})}{(K_{sp}+K_{sc}+K_d)}e^{-[K_m(K_{sp}+K_{sc}+K_d)]t}\right]dt.$$

Or

$$\{w_{imc}^2(0) - 4QG\}\lambda^2 - J^2 = 0$$

$$\lambda = \frac{J}{\sqrt{w_{imc}^2(0) - 4QG}}$$

λ takes on a positive value because as rises, the minimum efficiency loss also increases. From Equation (79):

$$A = \frac{\lambda w_{imc}(0) - J}{2\lambda Q} \quad (81)$$

Substituting the value of into the previous expression, we obtain:

$$A = \frac{\frac{w_{imc}(0)J}{\sqrt{w_{imc}^2(0)-4QG}} - J}{\frac{2QJ}{\sqrt{w_{imc}^2(0)-4QG}}},\ A = \frac{w_{imc}(0) - \sqrt{w_{imc}^2(0)-4QG}}{2Q} \quad (82)$$

The second-order condition for confirming minimization can be verified as demonstrated below:

$$\mathcal{L} = JA + \lambda[G - A(w_{imc}(0) - QA)]$$

The bordered Hessian matrix of the Lagrange function is displayed below:

$$BH = \begin{bmatrix} 0 & w_{imc}(0) - 2QA \\ w_{imc}(0) - 2QA & \frac{2QJ}{\sqrt{w_{imc}^2(0) - 4QG}} \end{bmatrix}$$

The determinant of the aforementioned matrix is negative because indicating that the efficiency loss has been minimized.