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Dynamics of Money Market and Monetary Policy

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Abstract

Objective: Contemporary research on monetary policy does not account for the loss/gain in efficiency during the adjustment of the market and the after-policy vis-a-vis pre-policy equilibrium in the money market. After a central bank exercises a monetary policy, the central bank's cost as a supplier of money rises to pre-policy cost plus the per unit money cost incurred due to monetary policy, which affects money supply and pushes the money market out of equilibrium. Demand and supply of money along with the interest rate follow certain adjustment mechanism until the final equilibrium arrives. The basis of adjustment is lack of coordination regarding decisions of consumers and suppliers of money at the prevailing interest rate. For the design of an optimal monetary policy, efficiency considerations both during the adjustment of the market as well as in final equilibrium are important to be taken care of. This research designs a dynamic money market model and derives an optimal monetary policy.

Methods: A perfectly competitive money market with five agents has been modeled. The equations maximizing their objectives have been derived and solved simultaneously to solve the model. An optimal monetary policy has been derived by minimizing the objective function of efficiency loss, i.e., supply or consumption of money lost in post-policy equilibrium vis-a-vis the pre-policy one, and the loss during the time market is adjusting subject to central bank's cost constraint.

Results: Derived mathematical expressions outline the optimal expansionary and contractionary monetary policies considering the adjustments in demand and supply over time.

Conclusion: The expressions are functions of demand, supply, and inventory curves' slopes as well as initial pre-policy equilibrium quantity of funds.

Keywords: money market, monetary policy, dynamic efficiency, interest adjustment path

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1 INTRODUCTION

A country's central bank formulates and implements monetary policy for optimally controlling supply of money to achieve economic goals leading to a sustainable growth of economy. Money supply can either expand or contract as

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a result of implementation of monetary policy depending on whether the policy is expansionary or contractionary respectively. Monetary policy operates through following tools: open market operations, direct lending to banks, bank reserve requirements, unconventional emergency lending programs, and managing market expectations (depending upon the credibility of the central bank). Monetary policy is based on steps of designing, announcing, and implementing policy by central bank, currency board, or any other authority having a mandate of controlling money market in economy. It involves managing the interest rates and money supply through changing interest rates, regulations involving exchange rate, purchase/sale of government bonds, and decisions regarding reserve ratio, etc., in order to influence growth, inflation, production, consumption, and liquidity, etc. Monetary policy formulation depends on a variety of macroeconomic indicators, such as economic growth in terms of GDP and various sectors, inflation, oil markets, international market prices, etc., to achieve various objectives, e.g., reduce inflation, maintain a steady economic growth, address unemplyment, etc. Monetary policy along with other economic measures can achieve objectives which a fiscal policy can also achieve, e.g., a stable economic growth.

Generally a country's central bank has the mandate to formulate and carry out the monetary policy with an objective of reducing inflation and unemployment, and allowing an inflation rate for a stable economic growth. The central bank is also responsible for managing interest rates for a long-term economic growth. It is responsible for regulating the financial sector and also liquidates commercial banks in crisis by acting as a lender of last resort.

An expansionary monetary policy is exercised if the aim is to increase the economic growth and expand the economic activity, in case the country faces a high unemployment rate during a crisis/recesssion or economic slowdown, etc. Lowering the interest rates through a variety of measures is also a part of an expansionary monetary policy, which promotes spending and discourages savings. This increases the money supply in the market, and may boosts spending on consumption as well as the investment goods. However, inflation can result from an expansionary monetary policy.

There are a number of monetary policy tools the central banks use. The open market selling and buying of short-term bonds, i.e., open market operations is one of those, which target the short-term interest rates, such as the federal funds rate. Through buying or selling assets, the central bank injects or removes money respectively into the banking system, and the banks either increase or decrease the interest rates, until the target set by central bank is achieved. Quantitative easing is also a process done through open market operations by the purchase of a specific quantity of assets to target specified increases in the money supply, so that the banks could provide loans more easily.

The central banks may also change the required collateral demanded by the central bank from the banks to fulfill its role as lender of last resort, i.e., the discount rate. The requirement of more collateral, and charging higher rates is a contractionary monetray policy and implies that the banks need to be careful of their lending with regard to risky loans. On the other hand, a lower requirement of collateral, and charging lower rates is an expansionary monetary policy and implies that the banks can make risky loans at lower rates, and can have lower reserves.

The central banks also have reserve requirements from the banks, i.e., an amount as a percentage of the deposits made by their customers, to be retained by them to make sure that their liabilities could be met. If the reserve requirement is low, the banks have more funds available to lend or purchase assets. If the requirement is high, it curtails the lending by banks leading to a lower money supply growth.

Besides standard monetary policies, unconventional monetary policies have also been popular in the central banks around the globe, i.e., USA, England, European Central Bank, and Japan, since the 2008 financial crisis, combining the quantitative easing, discount loans, and open market operations' aspects.

The central banks can also influence the market expectations through their public announcements regarding their upcoming policies. The statements and policy announcements of the central banks move the markets by making the investors guess about the future course of action of the central banks. Some central banks deliberately choose to keep the monetray policy unpredictable, so that the expectations do not start showing up in the market prices in advance, while others choose to be more predictable and open to avoid the volatility in the market due to unexpected policies. However, the effectiveness of the policy announcements is contingent upon central bank's credibility and other authorities which are responsible for designing the policy, making it public through announcement, and also implementing it.

Ideally speaking, the monetary institutions responsible for the monetary policy must be autonomous, to avoid any influence by the government, political elements, or any other institution. In practice, governments all over the world have varying degrees of interference in monetary institutions. The investors base their decisions on the announced monetary policy as well as the credibility of the monetary authority.

Monetarist theory came to the fore in the 1950s, drawing its cornerstone from the QTM and assuming that velocity in the quantity theory of money is generally stable, which implies that nominal income is largely a function of the money supply (Friedman[1]). Keynes rejected the quantity theory, both theoretically and as a tool of applied policy, in part arguing that velocity of money is unstable and not constant. QTM also assumed the absence of the trade-off between inflation and output (Keynes[2]). Keynesianism rationalized that prices are rigid and that the quantity of money adjusted rapidly. According to Hicks[3], the basic version of the IS_LM model assumes a fixed price level; and thus cannot be used to analyse inflation but output in the short run. Hicks IS/LM view of the Keynes's general theory was, however, contested empirically (Leijonhufvud[4]). Brunner and Meltzer[5] develops an alternative to the standard IS-LM framework. There are two asset markets and three prices-the prices of real assets, financial assets, and output. Friedman[6] argues that there is least agreement about the role that various instruments of policy can and should play in achieving the several goals. Evans[7] finds that money is not neutral in the long run if it is not in the short run, in particular, if growth is endogenous. If growth is exogenous, long-run neutrality is found. Goodfriend and King[8] describes the key features of the New Neoclassical Synthesis and its implications for the role of monetary policy. Bullard[9] support the long term neutrality of money. Sims[10] reviews Monetary Policy Rules, edited by John Taylor. Insolvent firms must not be recapitalized with taxpayer funds. Mankiw[11] shows that only monetary policy (unexpected) surprises would have a temporary effect on real variables. Monetarist upheld the principle of trade-off between inflation and output but reformulated the Philips curve in terms of real wage and not nominal wages (Gottschalk[12]). Palley[13] contrasts the new classical, neo-Keynesian, and Post Keynesian frameworks, thereby surfacing the differences. Empirical evidence on the use of New Keynesian models remains slim, and that practicality of theory is contested in part on grounds of absence of the role of money (Arestis and Sawyer[14]). Nogueira et al.[15] using annual data for 14 emerging and developed countries offer overall support for the traditional economic theory that monetary policy is neutral over the long-run. Schwartz[16] concludes that a systematic procedure for examining portfolios of the financial institutions needs to be followed to identify which are insolvent. White et al.[17] challenged Monetarism on grounds of technological developments and the instability of the money demand function. Galí[18] provides a rigorous graduate-level introduction to the New Keynesian framework and its applications to monetary policy, according to which the New Keynesian framework is the workhorse for the analysis of monetary policy and its implications for inflation, economic fluctuations, and welfare. D'Amico and King[19] estimates how much of the impact from the Fed's tightening cycle is to be felt in the U.S. economy in both absolute and relative terms. Storm[20] critically reviews the theoretical and empirical merits of three recent tweaks to the New Keynesian core: using the vacancy ratio as the appropriate measure of real economic activity; hammering on the considerable risk of an imminent wage-price spiral; and the resurrection of the non-linear Phillips curve. The paper concludes by drawing out sobering lessons concerning the art of paradigm maintenance as practiced by the "scientists of monetary policy".

Contemporary research on monetary policy does not account for the loss/gain in efficiency during the adjustment of the market and the after-policy vis-a-vis pre-policy equilibrium in the money market. After a central bank exercises a monetary policy, the central bank's cost as a supplier of money rises to pre-policy cost plus the per unit money cost incurred due to monetary policy, which affects money supply and pushes the money market out of equilibrium. Demand and supply of money along with the interest rate follow certain adjustment mechanism until the final equilibrium arrives. The basis of adjustment is lack of coordination regarding decisions of consumers and suppliers of money at the prevailing interest rate. For design of an optimal monetary policy, efficiency considerations both during the adjustment of the market as well as in final equilibrium are important to be taken care of. This research designs a dynamic money market model and derives an optimal monetary policy by minimizing the objective function of efficiency loss, i.e., supply or consumption of money lost in post-policy equilibrium vis-a-vis the pre-policy one, and also the loss during the time market is adjusting subject to central bank's cost constraint.

The remainder consists of the following sections: Section 2 connects agents in the money market to construct a dynamic money market model. Section 3 solves the model with an expansionary monetary policy. Section 4 derives an optimal expansionary monetary policy. Section 5 solves the model with a contractionary monetary policy. Section 6 derives an optimal contractionary monetary policy. Section 7 presents the summary of findings and conclusion.

2 THE MODEL

There is a perfectly competitive money market in equilibrium, and have five agents, with two of them as suppliers of money, i.e., the household, and the central bank, consumer of money for investment, i.e., a producer or firm, a middleman, i.e., a financial intermediary/commercial bank, and central bank acting as the government for formulating and implementing monetary policy. Central bank influences the interest rate through exercising monetary policy, however, in the role of supplier of money, takes the interest rate as given. The interest rate is also given for the household. If the money supply changes due to an exogenous shock, e.g., the introduction of debit/credit cards for payment instead of cash in the shopping malls, restaurants, etc., the interest rate cannot jump on its own to bring money market in final equilibrium. The commercial bank in the role of middleman varies the interest rate in its own benefit due to which the interest rate follows an adjustment path before bringing the money market in equilibrium. After the market attains final equilibrium, the commercial bank finds it optimal not to change interest rate and stay put. Money is supplied to commercial bank by suppliers, who holds the stock of money to supply it further to consumer, i.e., producer/firm, which borrows from financial intermediary at market interest rate. Money suppliers maximize their benefit; financial intermediary maximizes profit, i.e., the amount of revenue generated through lending money to the consumer/borrower minus the cost of holding stock of money, subject to the constraints; and the consumer of money, i.e., the firm/producer maximizes profit.

The adjustment of interest rates hinges on the idea that when an external shock disrupts the money market's equilibrium, sellers and buyers do not synchronize their decisions at the prevailing interest rate. To illustrate, let's envision a scenario: Initially, the money market is balanced, and the commercial bank holds a stable quantity of money. However, if an unexpected increase in money supply occurs, exceeding consumer demand at the current interest rate, a surplus accumulates with the bank. In response, the bank lowers interest rates to incentivize lower quantity of money supply by suppliers. Ultimately, a new equilibrium is achieved, characterized by a higher money quantity and lower interest rates than the initial state. This equilibrium is defined as follows:
(i) Money suppliers aim to maximize their utility or benefit; consumers, producers, or firms strive to maximize profit; and commercial banks aim to maximize profit by balancing the revenue earned from lending money to borrowers or consumers against the cost of holding money, within certain constraints (refer to Section 2 for further details).

(ii) In equilibrium, the demand for money matches the supply, and the commercial bank's money inventory remains unchanged.

In Section 3, the stability criterion of Routh–Hurwitz is discussed, which serves as the necessary and sufficient condition for equilibrium in a linear dynamical system. In a perfectly competitive money market, the financial intermediary or commercial bank operates as a price-taker, accepting the prevailing interest rate under market equilibrium. However, in times of market disequilibrium, the financial intermediary has an incentive to adjust interest rates until the money market reaches its final equilibrium, where it once again becomes a price-taker. When the central bank implements monetary policy, the market interest rate adjusts gradually rather than abruptly, facilitating the transition to a new equilibrium. This adjustment of interest rates is driven by endogenous decision-making processes involving households, the central bank, consumers of money, and financial intermediaries. During periods of steady-state equilibrium in the money market, both the financial intermediary and consumers borrow an amount of money equivalent to the supply provided by money suppliers in each time period. However, in the event of an expansionary monetary policy by the central bank—such as an increase in money supply—some portion of the supply may remain unutilized by consumers by the end of the policy period.

If money suppliers and the financial intermediary could instantly adjust money supply and market interest rates, and if the intermediary knew the new demand and supply patterns following the interest rate change, the commercial bank would set an optimal market interest rate to maximize profits and clear the money market. However, since this information is not available to the commercial bank or financial intermediary, they adjust interest rates based on their best estimate of the new market conditions, aiming to drive the market towards its final equilibrium. As the commercial bank reduces the interest rate, suppliers offer less money than before. The bank or intermediary continues lowering the interest rate until the market reaches its final equilibrium, which is gauged by the amount of unsold or unborrowed funds. Ultimately, the market settles into a new equilibrium, albeit with some efficiency losses during the adjustment period. These losses represent the unused funds during the money market adjustment due to monetary policy, and the total loss is the sum of the adjustment period loss plus or minus the loss or gain in the final equilibrium.

In mathematical terms, the first-order derivatives of the objective functions of all involved parties have been utilized to optimize their goals. These individual equations are then solved concurrently to derive a mathematical representation of their collective response. One simplifying assumption made is that the final equilibrium is not significantly distant from the pre-policy equilibrium. This implies that linearizing the demand and supply schedules is a reasonable approach. While Figure 1 illustrates that linearization is appropriate for the movement of equilibrium from point *a* to *b*, it's not realistic to assume linearity of the supply curve when the equilibrium shifts from point *a* to *c*. In such cases, a non-linear dynamical system, which is beyond the scope of this paper, would need to be considered.

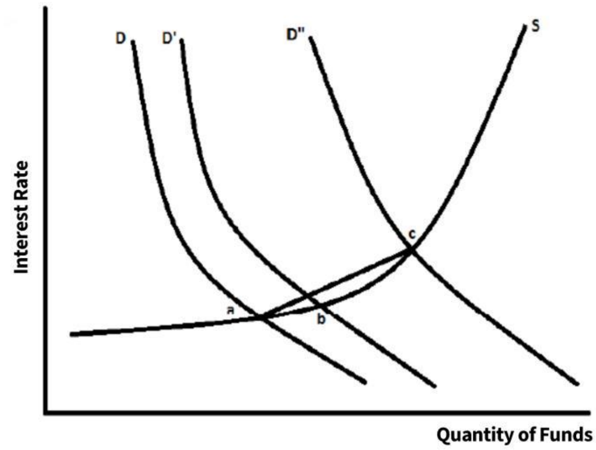

Figure 1. When is Linearity a Reasonable Assumption?

2.1 Financial Intermediary as Middleman

The financial intermediary or commercial bank acts as an intermediary, acquiring funds from suppliers to offer to consumers in exchange for profit through lending or selling. This intermediary maintains a money inventory, representing the difference between the funds purchased and sold over time. The inventory reflects the disparity between money supply and demand in the market. When the rates of supply and demand match, the inventory remains stable. Any change in the inventory suggests a discrepancy between supply and demand rates, indicating either a shift in one, the other, or both.

Figure 2 provides insight into the correlation between the money inventory, demand, supply, and interest rate. When there's a rightward shift in money supply with demand unchanged, the initial interest rate sees an increase in money inventory, followed by a decrease in the final equilibrium interest rate. Conversely, if money demand shifts rightward while supply remains constant, the initial interest rate witnesses a decrease in money inventory, leading to an increase in the final equilibrium interest rate. This suggests an inverse relationship between inventory change and interest rate change, all else being equal. If both money demand and supply shift in a way that keeps the inventory constant, the interest rate remains unchanged as well. Inventory serves as a unifying factor for demand and supply shocks, essentially acting as an inventory shock affecting the overall inventory. This observation underscores the inverse relationship between inventory change and corresponding interest rate change. The mechanism behind this phenomenon lies in the cost associated with holding an inventory of money by financial intermediaries. Larger inventories incur higher holding costs. In a scenario with equal demand and supply rates and no external shocks, the money market reaches a steady-state equilibrium, with the interest rate remaining unchanged.

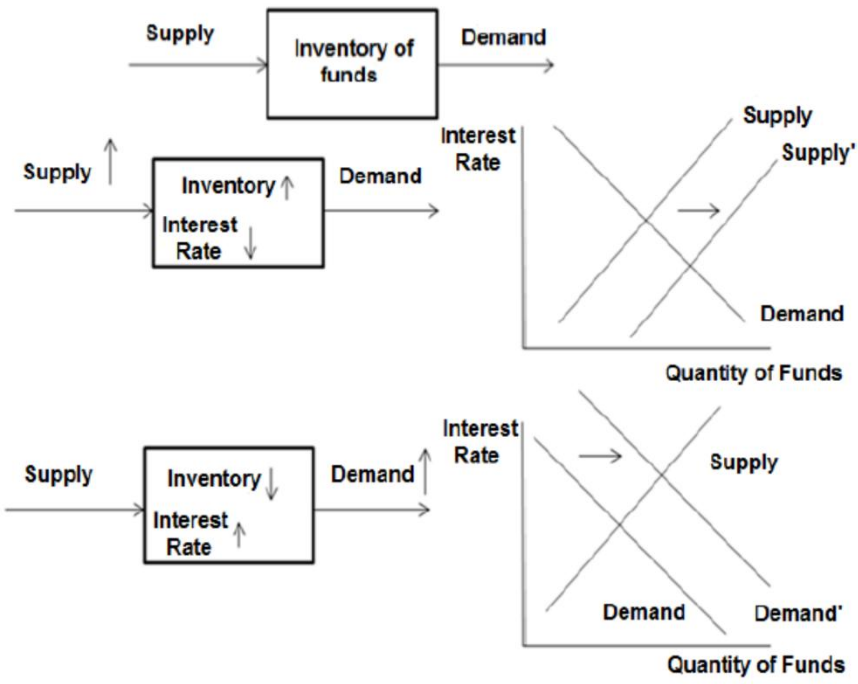

Figure 2. Movement of Interest Rate with Inventory of Funds.

Let's consider a situation where a policy lowers the marginal cost of savings, prompting households to increase the supply of money while demand remains unchanged. With the supply rate no longer matching the demand rate, the disparity accumulates as a surplus of funds held by financial institutions. To facilitate lending of these additional funds to borrowers, financial institutions would be inclined to decrease interest rates, stimulating demand along the demand curve. Eventually, in a perfectly competitive market, the interest rate will equate to the new marginal cost. However, the adjustment process in the market hinges on how financial intermediaries respond to changes in fund inventories. In this scenario, despite the decrease in the marginal cost of production, the marginal cost for financial intermediaries to hold an additional unit of funds has risen. To illustrate this model mathematically, we consider the profit maximization problem of the financial intermediary as follows:

2.1.1 One Time Period Problem

In this section, middleman's short-term (one time period) objective without consideration of profits in subsequent time periods is considered as follows:

$$\Pi = rq(r) - \varsigma(m(r,e)) \ (1)$$

Π =profit,

r market interest rate,

$q(r)$ quantity of funds sold (selling is in fact lending) at interest rate r,

m =inventory (quantity of funds the *middleman* holds),

e factors influencing inventory of funds other than market interest rate including interest rate at which middleman makes purchases from producers,

$\varsigma(m(r,e))$ cost as a function of inventory (increasing in inventory).

Taking derivative of eq. with respect to interest rate, we get:

$$rq^{'}(r) + q(r) - \varsigma^{'}(m(r,e))m_{1}^{'}(r,e) = 0 \ (2)$$

Financial intermediaries have a motivation to adjust interest rates only when the market is in transition. When the money market is balanced, altering interest rates away from marginal costs results in losses for the intermediary. The supply and demand do not align during this transitional phase, leading the market towards its ultimate equilibrium. Therefore, adjusting interest rates during this transitional period, which facilitates movement towards the final equilibrium, aligns with market dynamics. Changing interest rates during market equilibrium leads to lost business for intermediaries, unlike during the adjustment phase. During equilibrium, intermediaries encounter infinitely elastic demand, as illustrated below:

$$rq^{'}(r) + q(r) = \varsigma^{'}(m(r,e))m_1^{'}(r,e),$$

$$r\left[1 + \frac{1}{demand\ elasticity}\right] = \varsigma^{'}(m(r,e))\frac{m_1^{'}(r,e)}{q^{'}(r)}$$

In the scenario of infinitely elastic demand, the interest rate aligns with the marginal cost, as depicted in the preceding equation where the right-hand side represents the marginal cost. Suppose a supply shock occurs, diminishing the marginal cost of fund supply, causing the supply curve to shift downward. At this point, supply no longer matches demand at the initial equilibrium interest rate when the competitive market deviates from its steady state. Over time, the interest rate decreases, leading to the final equilibrium. However, there isn't an immediate leap in the interest rate; rather, the intermediary maintains a rate higher than the new marginal cost until realizing, through an accumulation of fund inventory, that the market's supply has increased. Consequently, adjusting the interest rate becomes necessary to adhere to the revised profit-maximizing condition. Similarly, a reverse supply shock leads to an eventual increase in the interest rate. In this situation, the intermediary maintains an interest rate below the new marginal cost until realizing, as fund inventory depletes, that the market's supply has decreased. Adjusting to the revised scenario's profit-maximizing condition necessitates an interest rate increase. During this adjustment phase, consumers benefit from lower rates until the intermediary raises them. In equilibrium, the market interest rate equals the sum of production and fund-holding marginal costs for the intermediary in the absence of policy intervention. This equilibrium signifies neither the intermediary nor the consumer gains an economic rent in a competitive market.

In a mathematical context, consider a scenario where a supply shock occurs, causing an increase in supply while demand remains unchanged. The financial intermediary's marginal cost for having an extra unit of funds, i.e., is higher at existing market interest rate on account of the term which is higher, i.e., the middleman has a higher cost for holding unutilized funds after supply shock. Meanwhile, the other term, $\varsigma^{'}(m(r,e))$, remains unchanged as the interest rate remains the same. This assumes that the financial intermediary's purchase rate remains constant, as fund producers are price-takers during the market adjustment period and only charge a fraction of the market interest rate to the intermediary. In discrete time, this is analogous to the financial intermediary maximizing profit in each period without considering future periods, taking the purchase interest rate as given and only determining the selling/lending market interest rate. At the current interest rate, the financial intermediary faces the following inequality:

$$\frac{\partial \Pi}{\partial r} = rq^{'}(r) + q(r) - \varsigma^{'}(m(r,e))m_1^{'}(r,e) < 0 \ (3)$$

This suggests that, following a supply shock, the financial intermediary needs to lower the interest rate to maintain profit maximization when acquiring additional funds. Currently, the producer of funds benefits from the short-term gains resulting from the reduced marginal cost of production, as the market interest rate remains unchanged until adjusted by the financial intermediary. A graphical representation of the optimal pairs of fund inventory and corresponding market interest rates, known as an *inventory curve* (similar to *demand* and *supply curves*), shows a downward slope, with interest rate on the y-axis and inventory on the x-axis.

2.1.2 Dynamic Problem

This part addresses the dynamic problem of financial intermediaries. They aim to maximize the present discounted value of an infinite series of future profits in such an environment. The present value at time zero is outlined as follows:

$$V(0) = \int_0^{\infty} [rq(r) - \varsigma(m(r,e))]e^{-\sigma t}dt \ (4)$$

The discount rate, control variable, and state variable are represented by and respectively. The maximization problem is expressed in mathematical notation as shown below:

$$\underset{\{r(t)\}}{Max} V(0) = \int_0^\infty [rq(r) - \varsigma(m(r,e))]e^{-\sigma t}dt,$$

while satisfying the following conditions:
$\dot{m}(t) = m_1^{'}(r(t), e(r(t), z))\dot{r}(t) + m_2^{'}(r(t), e(r(t), z))e_1^{'}(r(t), z)\dot{r}(t)$ (state equation illustrates the evolution of the state variable over time; are exogenous factors),
$m(0) = m_s$ (initial condition),
$m(t) \geq 0$ (the requirement that the state variable must be non-negative),
$m(\infty)$ free (terminal condition).

Below is the current-value Hamiltonian expression:

$$\widetilde{H} = r(t)q(r(t)) - \varsigma(m(r(t), e(r(t), z))) + \mu(t)\dot{r}(t)\begin{bmatrix} m_1^{'}(r(t), e(r(t), z)) + m_2^{'}(r(t), e(r(t), z)) * \\ e_1^{'}(r(t), z) \end{bmatrix} (5)$$

The conditions for maximization are provided as follows:
(i) maximizes for all t: $\frac{\partial \widetilde{H}}{\partial r} = 0$,
(ii) $\dot{\mu} - \sigma\mu = -\frac{\partial \widetilde{H}}{\partial m}$,
(iii) (this simply returns the state equation),
(iv) (the transversality condition).

The first and second conditions are outlined below:

$$\frac{\partial \widetilde{H}}{\partial r} = 0 \ (6)$$

and

$$\dot{\mu} - \sigma\mu = -\frac{\partial \widetilde{H}}{\partial m} = \varsigma^{'}(m(r(t), e(r(t), z))) \ (7)$$

In equilibrium, and expression boils down to the following (Appendix 1):

$$r(t)\left[1 + \frac{1}{demand\ elasticity}\right] = \varsigma^{'}(m(r(t), e(r(t), z)))\left\{\frac{m_1^{'}(r(t), e(r(t), z))}{q^{'}(r(t))} + \frac{m_2^{'}(r(t), e(r(t), z))e_1^{'}(r(t), z)}{q^{'}(r(t))}\right\}$$

This suggests that when demand is infinitely elastic, the interest rate equals the marginal cost. However, the marginal cost, depicted on the right-hand side of the equation, differs from that in a myopic problem because in dynamic considerations, the financial intermediary also considers the impact of the market interest rate on the purchase interest rate charged by producers. In the event of a positive supply shock, the marginal cost of holding an additional unit of funds in inventory increases for the financial intermediary, as the term is higher at that particular time with the existing interest rate. The term remains unchanged as the interest rate remains the same as before. At the current interest rate, the financial intermediary encounters the following inequality:

$$\frac{\partial \widetilde{H}}{\partial r} < 0$$

To fulfill the dynamic optimization condition, the financial intermediary needs to decrease the interest rate in order to include another unit of funds in its inventory. This indicates a negative correlation between the inventory of funds and the interest rate. The concepts of demand and supply are linked through inventory; when their rates match, market equilibrium is achieved. However, if there's a disparity between their rates and other agents do not respond to this change, the financial intermediary will continuously adjust the interest rate until market saturation occurs. This market behavior can be articulated as follows:
Change in interest rate $\propto$ change in market (funds) inventory.
R = change in interest rate.
$M = m - m_s$ = change in market inventory of funds,

m = funds inventory at time t,
m_s = funds inventory in steady state equilibrium.

$$\text{Input} - \text{output} = \frac{dm}{dt} = \frac{d(m - m_s)}{dt} = \frac{dM}{dt},$$
$$\text{or } M = \int (\text{input} - \text{output})dt.$$
$$\text{Change in interest rate} \propto \int (\text{supply rate} - \text{demand rate})dt, \text{or}$$
$$R = -K_m \int (\text{supply rate} - \text{demand rate})dt$$

The constant represents proportionality. The negative sign indicates that when the supply rate exceeds the demand rate, becomes negative, i.e., a decrease in the interest rate. This expression can also be represented as:

$$\int (\text{supply rate} - \text{demand rate})dt = -\frac{R}{K_m}, \text{or}$$
$$\int (w_i - w_0)dt = -\frac{R}{K_m} \; (8)$$
$$w_i = \text{supply rate},$$
$$w_0 = demand\ rate,$$
$$K_m = \text{dimensional constant}$$

At $t = 0$, when the supply rate equals the demand rate, indicating market equilibrium, Equation (9) can be written as:

$$\int (w_{is} - w_{0s})dt = 0 \; (9)$$

The subscript represents a value in a steady-state equilibrium, where $R = 0$. By subtracting equation from Equation (8), we get:

$$\int (w_i - w_{is})dt - \int (w_0 - w_{0s})dt = -\frac{R}{K_m}, \text{or}$$
$$\int (W_i - W_0)dt = -\frac{R}{K_m} \; (10)$$

$$\text{where } w_i - w_{is} = W_i = \text{change in supply rate},$$
$$w_0 - w_{0s} = W_0 = \text{change in demand rate}$$

R, and are deviation variables, indicating deviations from the steady-state equilibrium with initial values of zero. Equation can also be represented as:

$$R = -K_m \int W dt = -K_m M \; (11)$$

where If experiences a sudden change due to a factor unrelated to a change in fund inventory, this additional input can be incorporated into equation as follows:

$$R = -K_m \int W dt + B = -K_m M + B \; (11A)$$

In addition to the feedback effect of the interest rate, the funds inventory can also be subjected to an exogenous shock.

2.2 Suppliers

Different entities provide money. Below, we'll examine the dynamic problem of two key suppliers: households and the central bank.

2.2.1 Household

Households, acting as money suppliers, aim to maximize the present discounted value of their utility over all future periods. At the initial time (time zero), this objective can be expressed as the present value.

$$V(0) = \int_0^{\infty} U(x(t))e^{-\rho t}dt \quad (12)$$

The discount rate is represented by and stands for the control variable. The maximization problem takes the following form:

$$\underset{\{x(t)\}}{Max} V(0) = \int_0^{\infty} U(x(t))e^{-\rho t}dt,$$

under the constraints:

$\dot{a}(t) = r(t)a(t) + w(t) - p(t)x(t)$ (the state equation describes how the state variable changes over time). and are asset holdings (*state variable*), exogenous time path of wages and price of consumption respectively.

$a(0) = a_s$ (initial condition),

$a(t) \geq 0$ (the non-negativity constraint ensures that the state variable remains greater than or equal to zero), and

$a(\infty)$ free (terminal condition).

Below is the current-value Hamiltonian expression:

$$\widetilde{H} = U(x(t)) + \mu(t)[r(t)a(t) + w(t) - p(t)x(t)] \quad (13)$$

The conditions for maximization are provided as follows:

(i) maximizes for all t: $\frac{\partial \widetilde{H}}{\partial x} = 0$,

(ii) $\dot{\mu} - \rho\mu = -\frac{\partial \widetilde{H}}{\partial a}$,

(iii) (this simply returns the state equation), and

(iv) (the transversality condition).

The first and second conditions are outlined below:

$$\frac{\partial \widetilde{H}}{\partial x} = U^{'}(x(t)) - \mu(t)p(t) = 0 \quad (14)$$

and

$$\dot{\mu} - \rho\mu = -\frac{\partial \widetilde{H}}{\partial a} = -\mu(t)r(t) \quad (15)$$

In a situation where the interest rate, denoted as $r(t)$, increases, the condition stated above takes on the following structure:

$$\dot{\mu} - \rho\mu + \mu(t)r(t) > 0$$

Following an increase in the interest rate, the household must augment its asset holdings and consequently increase the money supply to fulfill dynamic optimization criteria. Let's denote as the market interest rate, and as a benchmark price of assets, such as the yield on an asset encompassing the savings cost, household profit, and financial intermediary profit. This serves as a reference for household decision-making, determining whether the money supply should be augmented in response to an interest rate hike. It functions as a parameter that may either remain constant temporarily or fluctuate over time, such as the cost of savings can change or remain fixed depending on the exogenous economic conditions.

$$W_{mp} = \text{Change in money supply by h. holds/public due to interest rate change},$$

The motivation for households to increase money supply lies in the difference between the market interest rate and the reference asset price c_p, i.e., Therefore,

$$W_{mp} \propto \alpha(r - c_p), \text{or}$$

The interest rate represents the return that the household receives from financial intermediaries, where is a constant less than 1. We can express the previous statement as:

$$W_{mp} = K_{sp}(r - c_p) \quad (16)$$

In a steady state equilbrium, or

$$0 = K_{sp}(r_s - c_{ps}) \ (17)$$

K_{sp} is proportionality constant. and are values in a steady state equilibrium. Subtracting Equation from Equation (16), we obtain:

$$W_{mp} = K_{sp}[(r - r_s) - (c_p - c_{ps})] =- K_{sp}(C_p - R) =- K_{sp}\varepsilon_p \ (18)$$

W_{mp}, C_p and are deviation variables.

2.2.2 Central Bank

The central bank aims to maximize the total future net benefits by discounting them to their present value. The initial present value is indicated below:

$$V(0) = \int_0^{\infty} [U_c(r(t)) - \varsigma_c(m(r(t)))]e^{-r_c t}dt \ (19)$$

$U_c(r(t))$ is the benefit for society, decreasing in interest rate, the higher the interest rate, the lower is the benefit. is the cost to the central bank, increasing in interest rate. The cost curve in relation to the interest rate is concave downwards, indicating a diminishing rate of decrease in slope.

The discount rate is represented by while and denote the control variable and state variable, respectively. The maximization problem is outlined as follows:

$$\underset{\{r(t)\}}{Max} V(0) = \int_0^{\infty} [U_c(r(t)) - \varsigma_c(m(r(t)))]e^{-r_c t}dt,$$

under the constraints:
$\dot{m}(t) =$ (the state equation describes how the state variable changes over time),
$m(0) = m_s$ (initial condition),
$m(t) \geq 0$ (the non-negativity constraint ensures that the state variable remains greater than or equal to zero), and
$m(\infty)$ free (terminal condition).
Below is the current-value Hamiltonian expression:

$$\widetilde{H} = U_c(r(t)) - \varsigma_c(m(r(t))) + \mu(t)m^{'}(r(t))\dot{r}(t) \ (20)$$

The conditions for maximization are provided as follows:
(i) maximizes for all t: $\frac{\partial \widetilde{H}}{\partial r} = 0$,
(ii) $\dot{\mu} - r_c\mu =- \frac{\partial \widetilde{H}}{\partial m}$,
(iii) (this simply returns the state equation), and
(iv) (the transversality condition).
The first and second conditions are outlined below:

$$\frac{\partial \widetilde{H}}{\partial r} = U_c^{'}(r(t)) - \varsigma_c^{'}(m(r(t)))m^{'}(r(t)) + \mu(t)m^{''}(r(t))\dot{r}(t) = 0 \ (21)$$

and

$$\dot{\mu} - r_c\mu =- \frac{\partial \widetilde{H}}{\partial m} = \varsigma_c^{'}(m(r(t))) \ (22)$$

If interest rate rises, the term goes down, and the central bank faces the following inequality:

$$\dot{\mu} - r_c\mu - \varsigma_c^{'}(m(r(t))) > 0$$

This suggests that the central bank should increase the money supply as the interest rate rises to fulfill the

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dynamic optimization condition. If the change in money supply is proportional to the change in interest rate, we can express it similarly to the household case, following the same logic.

$$W_{mc} =- K_{sc}(C_c - R) \ (23)$$

2.3 Consumer

A money consumer, such as a firm seeking funds for investment, aims to maximize the present discounted value of future profits, which can be expressed as the present value at time zero:

$$V(0) = \int_0^{\infty} \left[p(t)F\big(K(t), L(t)\big) - w(t)L(t) - r(t)I(t)\right]e^{-\varrho t}dt \ (24)$$

The firm's output price is denoted by while (labor), and (level of investment) represent the discount rate and control variables, respectively. stands for the state variable. The consumer's maximization problem can be stated as follows:

$$\underset{\{L(t),I(t)\}}{Max} V(0) = \int_0^{\infty} \left[p(t)F\big(K(t), L(t)\big) - w(t)L(t) - r(t)I(t)\right]e^{-\varrho t}dt,$$

under the constraints:

$\dot{K}(t) = I(t) - \delta K(t)$ (the state equation describes how the state variable changes over time),

$K(0) = K_0$ (initial condition),

$K(t) \geq 0$ (the non-negativity constraint ensures that the state variable remains greater than or equal to zero), and

$K(\infty)$ free (terminal condition).

Below is the current-value Hamiltonian expression:

$$\widetilde{H} = p(t)F\big(K(t), L(t)\big) - w(t)L(t) - r(t)I(t) + \mu(t)[I(t) - \delta K(t)] \ (25)$$

The conditions for maximization are provided as follows:

(i) and maximize for all t: and $\frac{\partial \widetilde{H}}{\partial I} = 0$,

(ii) $\dot{\mu} - \varrho\mu =- \frac{\partial \widetilde{H}}{\partial K}$,

(iii) (this simply returns the state equation),

(iv) (the transversality condition).

The first and second conditions are outlined below:

$$\frac{\partial \widetilde{H}}{\partial r} = U_c^{'}(r(t)) - \varsigma_c^{'}(m\big(r(t)\big))m^{'}\big(r(t)\big) + \mu(t)m^{''}(r(t))\dot{r}(t) = 0 \ (26)$$

$$\frac{\partial \widetilde{H}}{\partial L} = p(t)F_2^{'}\big(K(t), L(t)\big) - w(t) = 0 \ (27)$$

$$\frac{\partial \widetilde{H}}{\partial I} =- r(t) + \mu(t) = 0 \ (28)$$

and

$$\dot{\mu} - \varrho\mu =- \frac{\partial \widetilde{H}}{\partial K} =- \left[p(t)F_1^{'}\big(K(t), L(t)\big) - \delta\mu(t)\right] \ (29)$$

If increases (while keeping the level of investment the same as before), the firm encounters the following:

$$-r(t) + \mu(t) < 0$$

After an increase in the interest rate, the money consumer must decrease the level of investment to meet the dynamic optimization condition. Assuming that the change in the demand for money is proportional to the change in interest rate, denoted as R, this change can be expressed as follows: Change in money demand $\propto R$, or

$$W_d =- K_d R \ (30)$$

W_d represents the change in money demand due to a change in interest rate, denoted as R. This indicates that when is positive, is negative.

3 SOLUTION OF THE MODEL WITH AN EXPANSIONARY MONETARY POLICY

Expressions from Equation (11A), and respectively are stated below:

$$\frac{dR(t)}{dt} =- K_m W(t)$$
$$W_d(t) =- K_d R(t)$$
$$W_{mp} =- K_{sp}\left(C_p - R\right)$$
$$W_{mc} =- K_{sc}(C_c - R)$$

and

$$W(t) = W_m(t) - W_d(t),$$

if no exogenous demand or supply shock happens. The total money supply encompasses the supply from various sources including the central bank, households, and firms. It can be delineated as a combination of two primary sources: the central bank and the public (which includes households, firms, etc.), represented as follows:

$$W_m(t) =- K_{sp}\left[C_p(t) - R(t)\right] - K_{sc}[C_c(t) - R(t)] \ (31)$$

Subscripts and have been incorporated to distinguish between the public and the central bank respectively. We can merge the aforementioned expressions to form:

$$\frac{dR(t)}{dt} =- K_m[W_m(t) - W_d(t)]$$
$$=- K_m\left[- K_{sp}\left\{C_p(t) - R(t)\right\} - K_{sc}\{C_c(t) - R(t)\} + K_d R(t)\right]$$
$$=- K_m\left[- K_{sp}C_p(t) - K_{sc}C_c(t) + (K_{sp} + K_{sc} + K_d)R(t)\right]$$

Rearranging above expression gives:

$$\frac{dR(t)}{dt} + K_m(K_{sp} + K_{sc} + K_d)R(t) = K_m\left[K_{sp}C_p(t) + K_{sc}C_c(t)\right] \ (32)$$

The Routh-Hurwitz stability criterion which offers a necessary and sufficient condition for determining the stability of a linear dynamical system described by the above-mentioned differential equation requires $K_m(K_{sp} + K_{sc} + K_d) > 0$; and as and are all defined as positive numbers, this criterion holds. This guarantees that the money market reaches a new equilibrium each time it deviates from its initial equilibrium, thanks to the adjustment mechanisms in place.

In the money market, unlike the goods market, the central bank assumes the dual roles of the government and a fund producer. When the central bank lowers the interest rate on discount loans, its supply curve shifts downward, leading to a cost in its role as the government. To maintain clarity regarding the central bank's actions in the money market, it's crucial to distinguish between its roles as a fund producer and as a government entity. Let's denote the change in cost incurred by the central bank due to the reduction in interest rates on discount loans as A, while assuming the cost of the public's money supply remains constant. The above equation can thus be represented as:

$$\frac{dR(t)}{dt} + K_m(K_{sp} + K_{sc} + K_d)R(t) =- K_m K_{sc} A \ (33)$$

The solution is given by the following expression:

$$R(t) =- \frac{K_{sc}A}{(K_{sp}+K_{sc}+K_d)} + \frac{K_{sc}A}{(K_{sp}+K_{sc}+K_d)} e^{-\left[K_m(K_{sp}+K_{sc}+K_d)\right]t} \ (34)$$

$R(0) = 0$ (initial condition), and (final steady state equilibrium value). As a result of a monetary policy, the interest rate dynamics depends on parameters and A.

4 A DYNAMIC OPTIMAL EXPANSIONARY MONETARY POLICY

Depending on whether the monetary policy leans towards expansion or contraction, the post-policy equilibrium may see either efficiency gains or losses compared to the initial state. However, these aren't the sole determinants of efficiency shifts; adjustments in the money market during the transition to the final equilibrium also contribute to efficiency changes. When the monetary policy is enacted, the money supply either increases or decreases while demand remains constant at the initial interest rate, disrupting the money market's equilibrium. Subsequently, the adjustment of interest rates commences to reconcile the supply and demand and establish a new equilibrium in the money market. The post-policy equilibrium interest rate is influenced by the elasticity of demand and supply. As previously discussed, the alteration in total supply resulting from monetary policy is as follows:

$$W_m(0) = -K_{sp}\left[C_p(0) - R(0)\right] - K_{sc}[C_c(0) - R(0)] = K_{sc}A \quad (35)$$

as $R(0) = 0$

Due to monetary policy, the money supply either increases or decreases by $K_{sc}A$. Since demand remains constant, the money inventory also changes by $K_{sc}A$, aligning with the change in supply. Consequently, the market falls out of equilibrium, prompting market forces to drive the money market towards its final equilibrium by adjusting interest rates. As interest rate fluctuates, the demand and supply of money respond in kind through feedback mechanisms. A surplus in the money inventory indicates an excess supply compared to demand, and vice versa. Efficiency loss occurs when the money market is out of equilibrium, signifying that either the supply or consumption of money/funds is inefficient at that moment. This implies that the total efficiency loss during the adjustment of the money market is the cumulative difference between supply and demand at all points in time. After factoring in any efficiency loss in the post-policy equilibrium compared to the initial/pre-policy equilibrium, the total efficiency loss for expansionary and contractionary monetary policies respectively can be expressed as:

$$EL(\text{expansionary}) = \int_{-\infty}^{0} W_m(\infty)dt + \int_{0}^{\infty} [W_m(t) - W_d(t)]dt$$

$$= \int_{-\infty}^{0} W_m(\infty)dt + M(t) \quad (36)$$

$$EL(\text{contractionary}) = \left|\int_{t}^{\infty} W_m(\infty)dt + \int_{0}^{t} W_d(t)dt + M(t)\right|$$

$$= \left|\int_{0}^{t} W_d(t)dt + \int_{t}^{\infty} W_d(\infty)dt + M(t)\right|$$

$$= \left|\int_{0}^{\infty} W_d(t)dt + \int_{0}^{\infty} [W_m(t) - W_d(t)]dt\right|$$

$$= \left|\int_{0}^{\infty} W_m(t)dt\right| \quad (37)$$

With Expansionary Monetary Policy Cost Constraint:

Referring to Equation (31), the change in money supply resulting from a shift in the market interest rate is as follows:

$$W_m(t) = -K_{sp}\left[C_p(t) - R(t)\right] - K_{sc}[C_c(t) - R(t)]$$

The part of the supply coming from the central bank is and may be written as follows:

$$W_{mc}(t) = -K_{sc}[C_c(t) - R(t)],$$

$$w_{nmc}(t) - w_{imc}(0) = -K_{sc}[C_c(t) - R(t)],$$

$w_{imc}(0)$ represents the initial money supply provided by the central bank, while denotes the adjusted money supply after the central bank implements its monetary policy. The expression indicates the deviation from the initial equilibrium value over time. The cost of the monetary policy can be articulated as:

$$MPC = A[w_{imc}(0) + K_{sc}\{A + R(t)\}] \quad (38)$$

The challenge of minimizing efficiency loss while adhering to a constraint on monetary policy cost can be depicted as follows:

$$\min_{A} EL \text{ s.t. } MPC \leq G.$$

G represents the central bank's cost associated with implementing monetary policy. The variable to be chosen is the monetary policy itself, denoted as A, and the constraint becomes effective at time $t = 0$. The Lagrangian for the aforementioned problem is presented below:

$$\mathcal{L} = \int_{-\infty}^{0} W_m(\infty)dt + M(t) + \lambda\big[G - A[w_{imc}(0) + K_{sc}\{A + R(t)\}]\big]$$

$$= \int_{-\infty}^{0} \left[K_{sc}A - \frac{K_{sc}(K_{sp} + K_{sc})A}{(K_{sp} + K_{sc} + K_d)}\right] dt$$

$$-\frac{1}{K_m}\left[-\frac{K_{sc}A}{(K_{sp} + K_{sc} + K_d)} + \frac{K_{sc}A}{(K_{sp} + K_{sc} + K_d)} e^{-[K_m(K_{sp}+K_{sc}+K_d)]t} - K_m K_{sc} A\right]$$

$$+\lambda\left[G - A\left[w_{imc}(0) + K_{sc}\left\{A - \frac{K_{sc}A}{(K_{sp} + K_{sc} + K_d)} + \frac{K_{sc}A}{(K_{sp} + K_{sc} + K_d)} e^{-[K_m(K_{sp}+K_{sc}+K_d)]t}\right\}\right]\right]$$

$$= \int_{-\infty}^{0} \frac{K_{sc}K_d A}{(K_{sp} + K_{sc} + K_d)} dt - \frac{1}{K_m}\left[-\frac{K_{sc}A}{(K_{sp} + K_{sc} + K_d)} + \frac{K_{sc}A}{(K_{sp} + K_{sc} + K_d)} e^{-[K_m(K_{sp}+K_{sc}+K_d)]t} - K_m K_{sc} A\right]$$

$$+\lambda\left[G - A\left[w_{imc}(0) + K_{sc}\left\{A - \frac{K_{sc}A}{(K_{sp} + K_{sc} + K_d)} + \frac{K_{sc}A}{(K_{sp} + K_{sc} + K_d)} e^{-[K_m(K_{sp}+K_{sc}+K_d)]t}\right\}\right]\right]$$

Differentiating the Lagrangian with respect to yields the following first-order derivative:

$$A = - \frac{\lambda w_{imc}(0) - \left[\int_{-\infty}^{0} \frac{K_{sc}K_d}{(K_{sp}+K_{sc}+K_d)} dt - \frac{1}{K_m}\left[-\frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)} + \frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)} e^{-[K_m(K_{sp}+K_{sc}+K_d)]t} - K_m K_{sc}\right]\right]}{2\lambda K_{sc}\left[1 - \frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)} + \frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)} e^{-[K_m(K_{sp}+K_{sc}+K_d)]t}\right]} \quad (39)$$

The first-order derivative with respect to is as follows:

$$G - A\left[w_{imc}(0) + K_{sc}\left\{A - \frac{K_{sc}A}{(K_{sp}+K_{sc}+K_d)} + \frac{K_{sc}A}{(K_{sp}+K_{sc}+K_d)} e^{-[K_m(K_{sp}+K_{sc}+K_d)]t}\right\}\right] = 0 \quad (40)$$

Putting Equation into Equation (40), we get:

$$\lambda = \frac{J}{\sqrt{w_{imc}^2(0) + 4QG}}$$

$$\text{where } Q = K_{sc}\left[1 - \frac{K_{sc}}{(K_{sp} + K_{sc} + K_d)} + \frac{K_{sc}}{(K_{sp} + K_{sc} + K_d)} e^{-[K_m(K_{sp}+K_{sc}+K_d)]t}\right],$$

$$J = \int_{-\infty}^{0} \frac{K_{sc}K_d}{(K_{sp} + K_{sc} + K_d)} dt - \frac{1}{K_m}\left[-\frac{K_{sc}}{(K_{sp} + K_{sc} + K_d)} + \frac{K_{sc}}{(K_{sp} + K_{sc} + K_d)} e^{-[K_m(K_{sp}+K_{sc}+K_d)]t} - K_m K_{sc}\right]$$

λ is a positive quantity since an increase in leads to a rise in the minimum efficiency loss. Referencing Equation (39):

$$A = -\frac{\lambda w_{imc}(0) - J}{2\lambda Q} \quad (41)$$

By replacing with its value in the above expression, we get:

$$A =- \frac{w_{imc}(0) - \sqrt{w_{imc}^2(0) + 4QG}}{2Q} \ (42)$$

The second-order condition for minimization is detailed in the Appendix 1. Let's assume the government has \$1000 available to allocate towards monetary policy costs.

The initial value of the central bank's money supply is 100 discount loans of equal amount to the commercial banks, and value of all parameters, i.e., and is taken as one. After substituting these values into Equation (42), we obtain:

$$A =- \frac{100 - \sqrt{10000 + 4000}}{2} = 9.161,$$

where and at $Q = 1$. The monetary policy cost is given by Consequently, the optimal monetary policy entails the central bank reducing the interest rate on discount loans to the extent that, for each unit of discount loan, the government incurs a cost of only \$9.161.

5 SOLUTION OF THE MODEL WITH A CONTRACTIONARY MONETARY POLICY

The expressions from Equation (11A), and are as follows:

$$\frac{dR(t)}{dt} =- K_m W(t),$$
$$W_d(t) =- K_d R(t),$$
$$W_{mp} =- K_{sp}\left(C_p - R\right),$$
$$W_{mc} =- K_{sc}(C_c - R).$$

and

$$W(t) = W_m(t) - W_d(t),$$

if no exogenous demand or supply shock happens. The variable represents the total money supply, including contributions from the central bank, households, firms, etc. It can be expressed as a combination of two sources of money supply: the central bank and the public (which includes households, firms, etc.), as illustrated below:

$$W_m(t) =- K_{sp}\left[C_p(t) - R(t)\right] - K_{sc}[C_c(t) - R(t)] \ (43)$$

The subscripts and represent the public and the central bank respectively. By consolidating the preceding expressions, we can formulate:

$$\frac{dR(t)}{dt} =- K_m[W_m(t) - W_d(t)]$$
$$=- K_m\left[- K_{sp}\left\{C_p(t) - R(t)\right\} - K_{sc}\{C_c(t) - R(t)\} + K_d R(t)\right]$$
$$=- K_m\left[- K_{sp}C_p(t) - K_{sc}C_c(t) + (K_{sp} + K_{sc} + K_d)R(t)\right].$$

Rearranging the above expression gives:

$$\frac{dR(t)}{dt} + K_m(K_{sp} + K_{sc} + K_d)R(t) = K_m\left[K_{sp}C_p(t) + K_{sc}C_c(t)\right] \ (44)$$

When the central bank raises the interest rate on discount loans, its supply curve shifts to the left, generating revenue for the government. Let's denote the change in the central bank's costs resulting from the increased interest rate on discount loans as while assuming that the cost of the public's money supply remains unchanged. This equation can thus be expressed as follows:

$$\frac{dR(t)}{dt} + K_m(K_{sp} + K_{sc} + K_d)R(t) = K_m K_{sc} A \ (45)$$

The solution is given by the following expression:

$$R(t)=\frac{K_{sc}A}{(K_{sp}+K_{sc}+K_d)}-\frac{K_{sc}A}{(K_{sp}+K_{sc}+K_d)}e^{-[K_m(K_{sp}+K_{sc}+K_d)]t} \quad (46)$$

$R(0)=0$ (initial condition), and (final equilibrium value in a steady state).

6 A DYNAMIC OPTIMAL CONTRACTIONARY MONETARY POLICY

Eq. states the following:

$$R(t)=-K_mM(t)+B$$

Applying the initial conditions allows us to ascertain the specific value of (for contractionary monetary policy) as follows:

$$R(0)=-K_mM(0)+B,$$
$$0=K_mK_{sc}A+B,$$
$$B=-K_mK_{sc}A.$$

After substituting the aforementioned expression into Equation (11A), it undergoes transformation to:

$$R(t)=-K_mM(t)-K_mK_{sc}A,\text{or}$$
$$M(t)=-\frac{1}{K_m}[R(t)+K_mK_{sc}A] \quad (47)$$

With Contractionary Monetary Policy Revenue Constraint:

Referring to Equation the alteration in money supply resulting from a shift in the market interest rate is given by:

$$W_m(t)=-K_{sp}[C_p(t)-R(t)]-K_{sc}[C_c(t)-R(t)]$$

The part of the supply coming from the central bank is and can be written as follows:

$$W_{mc}(t)=-K_{sc}[C_c(t)-R(t)],$$
$$w_{nmc}(t)-w_{imc}(0)=-K_{sc}[C_c(t)-R(t)],$$

where represents the initial money supply from the central bank, and denotes the new money supply after the central bank implements the monetary policy. $W_{mc}(t)=w_{nmc}(t)-w_{imc}(0)$, indicating the deviation from the initial steady-state equilibrium value. The revenue gained by the government due to a contractionary monetary policy can be formulated as:

$$MPR=A[w_{imc}(0)-K_{sc}\{A-R(t)\}] \quad (48)$$

The problem of minimizing efficiency loss while adhering to a monetary policy constraint is outlined as follows:

$$\min_T EL \quad \text{s.t.} \quad MPR\geq G$$

G represents the income generated by the government through the implementation of monetary policy. The decision variable is the monetary policy denoted as A, and the constraint is binding at time $t=0$. The Lagrangian expression for this problem can be formulated as follows:

$$\mathcal{L}=-\int_0^\infty [K_{sp}\{C_p(t)-R(t)\}+K_{sc}\{C_c(t)-R(t)\}]dt+\lambda[G-A[w_{imc}(0)-K_{sc}\{A-R(t)\}]]$$

$$=\int_0^\infty\left[-K_{sc}A+\frac{K_{sc}(K_{sp}+K_{sc})A}{(K_{sp}+K_{sc}+K_d)}-\frac{K_{sc}(K_{sp}+K_{sc})A}{(K_{sp}+K_{sc}+K_d)}e^{-[K_m(K_{sp}+K_{sc}+K_d)]t}\right]dt$$

$$+\lambda\left[G-A\left[w_{imc}(0)-K_{sc}\left\{A-\frac{K_{sc}A}{(K_{sp}+K_{sc}+K_d)}+\frac{K_{sc}A}{(K_{sp}+K_{sc}+K_d)}e^{-[K_m(K_{sp}+K_{sc}+K_d)]t}\right\}\right]\right]$$

The derivative of the Lagrangian with respect to yields the following expression:

$$A=\frac{\lambda w_{imc}(0)-\int_0^\infty\left[-K_{sc}+\frac{K_{sc}(K_{sp}+K_{sc})}{(K_{sp}+K_{sc}+K_d)}-\frac{K_{sc}(K_{sp}+K_{sc})}{(K_{sp}+K_{sc}+K_d)}e^{-[K_m(K_{sp}+K_{sc}+K_d)]t}\right]dt}{2\lambda K_{sc}\left[1-\frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)}+\frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)}e^{-[K_m(K_{sp}+K_{sc}+K_d)]t}\right]} \quad (49)$$

First order derivative with respect to is as given below:

$$G - A\left[w_{imc}(0) - K_{sc}\left\{A - \frac{K_{sc}A}{(K_{sp}+K_{sc}+K_d)} + \frac{K_{sc}A}{(K_{sp}+K_{sc}+K_d)}e^{-[K_m(K_{sp}+K_{sc}+K_d)]t}\right\}\right] = 0 \ (50)$$

Putting Equation into Equation (50), we obtain:

$$\lambda = \frac{J}{\sqrt{w_{imc}^2(0) - 4QG}},$$

$$\text{where } Q = K_{sc}\left[1 - \frac{K_{sc}}{(K_{sp} + K_{sc} + K_d)} + \frac{K_{sc}}{(K_{sp} + K_{sc} + K_d)}e^{-[K_m(K_{sp}+K_{sc}+K_d)]t}\right],$$

$$J = \int_0^\infty \left[-K_{sc} + \frac{K_{sc}(K_{sp} + K_{sc})}{(K_{sp} + K_{sc} + K_d)} - \frac{K_{sc}(K_{sp} + K_{sc})}{(K_{sp} + K_{sc} + K_d)}e^{-[K_m(K_{sp}+K_{sc}+K_d)]t}\right]dt.$$

or

$$\{w_{imc}^2(0) - 4QG\}\lambda^2 - J^2 = 0$$

λ takes on a positive value because as rises, the minimum efficiency loss also increases. From Equation (49):

$$A = \frac{\lambda w_{imc}(0) - J}{2\lambda Q} \ (51)$$

Substituting the value of into the previous expression, we obtain:

$$A = \frac{w_{imc}(0) - \sqrt{w_{imc}^2(0) - 4QG}}{2Q} \ (52)$$

The second-order condition for minimization is presented in the Appendix 1.

7 CONCLUSION

When a government exercises an expansionary/contractionary monetary policy, the central bank's supply curve shifts downward/upward, which affects the money supply in the market and equilibrium no longer holds. Over time, the money supply and demand, along with the interest rate, dynamically adjust to guide the market towards its final equilibrium. Throughout this adjustment process and in the final equilibrium, there are efficiency gains or losses compared to the initial equilibrium in the money market. It's crucial to consider these efficiency changes, including those during the adjustment period, when devising an optimal monetary policy. Equations and outline the optimal expansionary and contractionary monetary policies, respectively, taking into account the adjustments in demand and supply over time. The expressions are functions of demand, supply and inventory curves' slopes as well as initial pre-policy equilibrium quantity of funds.

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Conflicts of Interest
The authors declared no conflict of interest.

Author Contribution
Ahmed MA conceived the main idea of the paper, planned on methodology, did literature review, and sketched outlines of the models. Nawaz N worked on details of the models and derived mathematical results. Both authors jointly prepared the working draft of the article, proofread, and agreed on the final draft for submission to the journal.

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