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Research Article

Quaternion Solution to the Problem of Optimal Control for Spacecraft'S Spatial Reorientation With Use of a Combined Quality Criterion Taking Into Account Energy Costs

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Abstract

Objective: This research focuses on economically controlling spacecraft motion to minimize energy costs during reorientation. It constructs an optimal control method that considers both control forces and rotational kinetic energy, aiming for efficient spatial reorientation.

Methods: Utilizing the quaternion method and Pontryagin's maximum principle, the study devises a restricted control for optimal spacecraft turns. Analytical solutions are derived from differential equation relating orientation quaternion and angular velocity, with numerical simulations used for verification.

Results: The study presents a solution to the optimal control synthesis problem for spacecraft reorientation, optimizing for minimal energy costs. Key properties of optimal solutions are formulated analytically, aiding in determining optimal control algorithm parameters. Mathematical modeling illustrates the process and practical feasibility of the designed method for attitude control.

Conclusion: This research provides a comprehensive solution to spacecraft orientation optimal control, particularly beneficial for spacecraft equipped with electric-jet engines. The explicit control law improves efficiency and economizes spacecraft motion during orbit flight, contributing to smoother, continuous functions of time for both control functions and phase variables. The study's relevance lies in addressing the cost-effectiveness of spacecraft motion control, with potential for further research on optimal control with additional restrictions.

Keywords: spacecraft's spatial attitude, quaternion, angular velocity, criterion of quality, maximum principle

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1 INTRODUCTION

Problem of spacecraft reorientation into given angular position was solved. The solution method and the formalized description of spacecraft's rotational motion kinematics are based on the quaternion models for description of rotational motion of solid body[1]. Spatial reorientation is a moving the spacecraft axes from one known attitude into another given angular position in finite time T. Angular attitude of spacecraft's coordinate system is determined relative to a chosen reference basis. We considered the version of spatial turn when inertial coordinate system is reference system as often encountered case.

Much of papers have been dedicated to investigating a controlled rotation of solid body and problem of optimal control for spacecraft attitude in different statements and using different methods of solving[1-26]. For example, some authors propose the synthesis of optimal control based on the method of analytical design of optimal controllers[2], others use the concept of inverse problems of dynamics to obtain smooth controls for the implementation of the spatial rotation of the spacecraft, when the program trajectory is sought in the class of polynomials of a given degree, the coefficients of which are determined by the known values of the phase variables at the boundary points of the trajectory[3]. Special attention was paid to the problems of optimal control[2,4-24]. Optimization methods can also be different. In particular, solutions to the problem of reorientation of solid body of various configurations, based on the Pontryagin's maximum principle, were considered in papers[9-24]. The time-optimal control is very importantly, therefore this maneuver is popular and interesting[5-12]. Other classical criteria for the quality of the control process were used previously (minimum fuel consumption[17], minimum energy consumption[12,17], etc.). Kinematic problems of rotation were considered in more detail[13-16]. Dynamic problems of optimal control are of particular interest and, at the same time, certain difficulties in solving the boundary value problem of a rotation; in some particular cases of control over a fixed time, the boundary two-point rotation problem is solved by the method of separation of variables[17]. Special regime of control for spacecraft rotation was examined also[18]. Specific features of attitude control for a spacecraft with inertial actuators (in particular, the gyrodins) have been researched earlier[25,26]. The patented method can be used in control system of a spacecraft, controlled by the gyrodins (or other inertial actuators)[27].

Earlier, planar rotations[5,8], relay controls for a turn[1,5-9,12], or the algorithms without optimization[3] for finding the smooth control functions were investigated. It is necessary to select out especially problems of the time-optimal turn[1,5-12], control problems for an axi-symmetric rigid body[10-13,18,20-22], and also kinematical problems of optimal turn[13-16]. Here, the dynamic problem of optimal control of a turn with the restricted control and the combined criterion of quality reflecting total energy costs (energy contribution of the control torques and integral costs of kinetic energy) is considered and solved in analytic form.

Analytical solution to an optimal turn problem in a closed form is of great practical interest because such solution allows the finished laws of the programmed control and variation of an optimal trajectory of spacecraft's motion to be applied onboard. However, it is extremely difficult to obtain them for bodies (spacecrafts) with an arbitrary dynamic configuration. Some solutions (including analytical ones) were obtained for spherical[1,19] and dynamically symmetric bodies[10-13,20-22]. But analytical solution to the problem of three-dimensional turn with arbitrary boundary conditions (for angular position of a spacecraft) was not found for solid body with arbitrary distribution of mass. We know certain particular cases when general problem of a turn is solved[1,12,19]. Consequently, we can use only numerical methods (for approximate solution to this problem).

Below, we solve the problem of reorientation with new index of quality and the restricted control. In this article, the adopted functional of quality characterizes the energy costs as combination of costs of control resources and rotation energy. Its minimization is very important task in practice of spacecraft flight. For the time being, the issues of cost-effectiveness remain relevant for spacecraft motion control. A finding and investigating the optimal control of spacecraft reorientation from initial angular position into given spatial position (with respect to the chosen combined indicator) is the purpose of our research.

Optimal control problems for spacecraft orientation with use of the combined criterions of optimality were investigated earlier[21-23]. However, in contrast to the published papers, the quality criterion used by us provides smooth control functions (it is firstly) and spacecraft motion with kinetic energy of rotation which is minimum for given time of a turn (it is secondly). Additionally, in our problem of optimal reorientation, the restricted control torque acts to a spacecraft for its rotation. Also, it should be noted that solution obtained in the presented research can be applied for any type of a spacecraft (but not only for spacecrafts with inertial actuators[25,26]), in contrast to the previous researches.

2 MATERIALS AND METHODS

2.1 Formulation of Optimal Control Problem

Spatial rotation of a spacecraft around the center mass is described by the quaternions (Rodrigues-Hamilton parameters). Motion of the body-fixed basis E relative to the reference basis I will be given by a quaternion Λ[1] (a body-fixed basis

E is formed by the principal central axes of ellipsoid of inertia). Without loss of generality, it is assumed that basis I is inertial. The angular positions of initial and final attitude of a spacecraft with respect to a reference basis I are given by the quaternions Λ_{in} and Λ_f, respectively. A following kinematical Equation (1) holds[1]:

$$2\dot{\Lambda} = \Lambda \circ \omega \quad (1)$$

where ω is the absolute angular velocity vector; the symbol ∘ is the sign of multiplication of quaternions[1].

Rotary motion of a spacecraft (as solid body) is described by the following Equations (2)[1]:

$$J_1\dot{\omega}_1 + (J_3 - J_2)\omega_2\omega_3 = M_1$$
$$J_2\dot{\omega}_2 + (J_1 - J_3)\omega_1\omega_3 = M_2 \quad (2)$$
$$J_3\dot{\omega}_3 + (J_2 - J_1)\omega_1\omega_2 = M_3$$

where ω_i are angular velocities about main central axes of the spacecraft's inertia ellipsoid; J_i are the main central moments of inertia of a spacecraft, M_i are the projections of control torque M on the main central axes of spacecraft's inertia ellipsoid. Control of spacecraft motion relative to the center of mass is carried out by changing a torque M (so-called the controlling torque).

For simplicity, the quaternion Λ determining current position of a spacecraft is assumed a normalized quaternion, ($\|\Lambda\|$=1). Further, we assume that a region of admissible values of a vector M is similar to spacecraft's inertia ellipsoid[9,12]:

$$\frac{M_1^2}{J_1} + \frac{M_2^2}{J_2} + \frac{M_3^2}{J_3} \le u_0^2 \quad (3)$$

where $u_0>0$ determines the control capabilities of spacecraft's attitude control system.

In practice, problems are interesting when angular velocity of a spacecraft at initial and final instants of time is zero. We write the boundary conditions for the dynamical system (1)-(3):

$$\Lambda(0)=\Lambda_{in},\ \omega(0)=0 \quad (4)$$

$$\Lambda(T)=\Lambda_f,\ \omega(T)=0 \quad (5)$$

where T is the time of the end the maneuver. The quaternions Λ_{in} and Λ_f satisfy the condition $\|\Lambda_{in}\| = \|\Lambda_f\| =1$. We know that the quaternions Λ and - Λ correspond to same position of a solid, therefore we only consider the problems in which $\Lambda_f \neq \pm\Lambda_{in}$.

We suppose that angular motion of a spacecraft is governed by attitude system generated the torques about three main central axes of inertia. Optimum is control that provides minimal value of the following sum

$$G = k_0\int_0^T (M_1^2/J_1 + M_2^2/J_2 + M_3^2/J_3)dt + \int_0^T (J_1\omega_1^2 + J_2\omega_2^2 + J_3\omega_3^2)dt \quad (6)$$

where $k_0>0$ is a constant positive coefficient ($k_0 \neq 0$).

The problem of optimal control is formulated in the following statement: a spacecraft must be rotated from the state Equation (4) into state Equation (5) according to the Equations (1) and (2) under condition Equation (3) so that the indicator Equation (6) is minimum (time T is given). Solution M(t) is sought within the class of piecewise continuous functions of time.

The adopted quality criterion Equation (6) distinguishes the proposed optimization problem from the other problems (considered earlier) by the form of the functional to be minimized with accounting for constraint Equation (3). Presence of integral of rotation energy limits energy of rotation E_k during optimal turn. We must note that the declared rotation maneuver can not be implemented for any values of Λ_{in}, Λ_f and J_1, J_2, J_3, k_0, u_0 since the time T is fixed, and control M is bounded by the requirement Equation (3). The problem of optimal rotation of a spacecraft with the restricted control is still relevant when the quality of control process is characterized by the index Equation (6).

Notice that optimization of rotations with minimal costs on the base of indicator Equation (6) can be useful for spacecrafts with attitude system which use electric-jet engines or electric-rocket engines (ERJ), because when controlled by an ERJ (in particular, ion engines), the first integral in index Equation (6) is proportional to the consumed electric power (for ERJ, engine's thrust is directly proportional to the consumed electric current[28], and the control torque is proportional to the installation arm of ERJ-engine).

2.2 Method for Solving the Problem of Optimal Rotation of Spacecraft

The described problem Equations (1)-(6) is a dynamic problem of optimal turn of solid body[12], in which M_i are control functions ($i = 1, 3$). We solve the formulated problem Equations (1)-(6) using the Pontryagin's maximum principle[29]. Quaternion variables are very effective mathematical technique applied successfully in many fields of physics sciences[30-32] (but not only for research of the controlled motion of solid body); it is the most widely applied mathematical tool. Also, we applied the mathematical modeling and numerical simulation for verification and confirming the practical feasibility of a designed mode of spacecraft turn. Solving the systems of differential equations was carried out by the method of successive approximations (in particular, the sweep method or the shooting algorithm). Two-point boundary-value problem (a boundary-value problem of a turn) we solve using the method of iteration; an integrating of equations uses different known numeric methods.

According to the maximum principle, we introduce the conjugate variables φ_i that correspond to the projections of angular velocity ω_i ($i=\overline{1,3}$). Since the criterion of quality Equation (6) does not contain elements of quaternion Λ, we use the following variables r_i[19], replacing the conjugate functions ψ_j, which correspond to components λ_j of quaternion Λ ($i=\overline{1,3}$, $j = 0, 3$):

$$r_1 = (\lambda_0\psi_1 + \lambda_3\psi_2 - \lambda_1\psi_0 - \lambda_2\psi_3)/2$$
$$r_2 = (\lambda_0\psi_2 + \lambda_1\psi_3 - \lambda_2\psi_0 - \lambda_3\psi_1)/2$$
$$r_3 = (\lambda_0\psi_3 + \lambda_2\psi_1 - \lambda_3\psi_0 - \lambda_1\psi_2)/2$$

Optimal functions r_i and the vector **r** formed by r_i satisfy the equations

$$\dot{r}_1 = \omega_3 r_2 - \omega_2 r_3, \dot{r}_2 = \omega_1 r_3 - \omega_3 r_1$$
$$\dot{r}_3 = \omega_2 r_1 - \omega_1 r_2, \dot{r} = r \times \omega \ (7)$$

(the symbol×means the cross product of vectors).

Let us write the Hamilton-Pontryagin function for the problem of optimal control Equations (1)-(6)

$$H=-k_0(M_1^2/J_1+M_2^2/J_2+M_3^2/J_3)-(J_1\omega_1^2+J_2\omega_2^2+J_3\omega_3^2)+\varphi_1(M_1+(J_2-J_3)\omega_2\omega_3)/J_1+$$
$$+\varphi_2(M_2+(J_3-J_1)\omega_1\omega_3)/J_2+\varphi_3(M_3+(J_1-J_2)\omega_1\omega_2)/J_3+\omega_1 r_1+\omega_2 r_2+\omega_3 r_3 \ (8)$$

The equations for the conjugate functions φ_i are obtained from the formulas[29]

$$\dot{\phi}_i = -\frac{\partial H}{\partial \omega_i}$$

And the adjoint system of equations is

$$\dot{\phi}_1 = 2J_1\omega_1 + \omega_3\phi_2(J_1 - J_3)/J_2 + \omega_2\phi_3(J_2 - J_1)/J_3 - r_1$$
$$\dot{\phi}_2 = 2J_2\omega_2 + \frac{\omega_3\phi_1(J_3 - J_2)}{J_1} + \frac{\omega_1\phi_3(J_2 - J_1)}{J_3} - r_2 \ (9)$$
$$\dot{\phi}_3 = 2J_3\omega_3 + \omega_2\phi_1(J_3 - J_2)/J_1 + \omega_1\phi_2(J_1 - J_3)/J_2 - r_3$$

The Hamiltonian H is compiled without taking into account the condition $||\Lambda||=1$ because the equality $\|\Lambda(0)\|=1$ and $\|\Lambda(t)\|=$const. The Equations (7) show that the vector r is motionless relative to inertial basis[19] (also $|r|=\text{const}\neq 0$). The initial Λ_{in} and finish Λ_f positions determine concrete solution r(t) of the system Equation (7). Moreover $r(0)\neq 0$ (otherwise $r_1=r_2=r_3\equiv 0$ and the further solving the problem makes no sense). Optimal function r(t) is calculated with use of the quaternion Λ(t)[12]

$$r=\tilde{\Lambda}\, c_E \Lambda, \text{ and } c_E = \text{const} = \Lambda_{in} r(0)\, \tilde{\Lambda}_{in}$$

where $\tilde{\Lambda}$ is the conjugate quaternion relative to the quaternion Λ[1].

Problem of searching the optimal control has been reduced to solving a system of equations of spacecraft's motion Equations (1), (2), (7) and (9) under condition that a found control itself is chosen by maximization of

Hamiltonian Equation (8) under restriction Equation (3). System of Equation (7) that determines a behavior of vector **r** relative to body-fixed basis replaces a conjugate system of equations for variables ψ_j.

2.3 Application of the Maximum Principle for Determining a Structure of Optimal Control

To find the control function M(t) and optimal vector r, we must know the conditions of maximum for Hamiltonian H. We assume $u_i = M_i/\sqrt{J_i}$ and $n_i = \phi_i/\sqrt{J_i}$ (i=1, 3). Passing to the new controls u_i and auxiliary variables n_i, we can rewrite the function H in the following form

$$H = u \cdot n - k_0|u|^2 + H_{inv} = |n||u|\cos\delta - k_0|u|^2 + H_{inv}$$

where H_{inv} does not explicitly depend on control function M_i; u and n are the vectors formed by variables u_i and n_i; δ is an angle formed by the vectors u and n (the sign "·" denotes scalar product of vectors).

The function H is maximal if δ=0. If $|u|<u_0$, then the maximum of Hamiltonian H in an argument u is inside the interval [0, u_0] and coincides with the local maximum. From formula (8) we see that H is quadratic function of control vector M, and the local maximum is determined by the necessary conditions of extremum $\partial H/\partial M_i = 0$. Using these requirements, we calculate optimal values of control functions

$$M_i=\varphi_i/(2k_0) \quad (10)$$

The control Equation (10) satisfies condition Equation (3) if $\phi_1^2/J_1 + \phi_2^2/J_2 + \phi_3^2/J_3 \le 4k_0^2u_0^2$ only.

If $\phi_1^2/J_1 + \phi_2^2/J_2 + \phi_3^2/J_3 > 4k_0^2u_0^2$, then an extremum of the Hamiltonian H is outside the interval $0\le|u|\le u_0$, i.e. maximum of function H is achieved when the desired control reaches the boundary $|u|=u_0$, and the vectors u and n have same direction. Hence, the optimal control is

$$M_i=\begin{cases} \varphi_i/(2k_0), \text{ if} \\ \phi_1^2/J_1 + \phi_2^2/J_2 + \phi_3^2/J_3 \le 4k_0^2u_0^2 \\ \dfrac{u_0\phi_i}{\sqrt{\phi_1^2/J_1+\phi_2^2/J_2+\phi_3^2/J_3}}, \text{ if} \\ \phi_1^2/J_1 + \phi_2^2/J_2 + \phi_3^2/J_3 > 4k_0^2u_0^2 \end{cases} \quad (11)$$

Thus, the structure of optimal control is determined. The dependences Equation (11) together with the Equations (7) and (9) are the necessary conditions of optimality. Problem of designing optimal control consists in solving a system of equations of rotation Equations (1), (2), (7) and (9) under condition Equation (11) for control torque M. Solution of the system of equations exists, and such solution is single (unique). The vector r(0) should be such that, as a result of integrating the Equations (1), (2), (7), (9) and (11) with the initial conditions $\Lambda(0)=\Lambda_{in}$, for the trajectory of motion $\Lambda(t)$ the requirement $\Lambda(T)=\Lambda_f$ holds.

After introduce the parameter $r_0=|r(t)|$=const≠0, we transfer to a normalized vector p=r/|r| (for simplicity), and $p_i=r_i/r_0$, |p|=1. For the vector p and its components p_i, we consider the following equations

$$\dot{p}=-\omega\times p,\ \dot{p}_1 = \omega_3 p_2 - \omega_2 p_3$$

$$\dot{p}_2 = \omega_1 p_3 - \omega_3 p_1,\ \dot{p}_3 = \omega_2 p_1 - \omega_1 p_2 \quad (12)$$

The closed system of Equations (1), (2), (7), (9) and (11) allows one to find the optimal control. Problem of constructing the optimal control of spacecraft rotation is reduced to solving a system of equations of spacecraft's motion Equations (1) and (2), conjugate Equations (9) and (12) with equalities $r_i=r_0p_i$ and presence of the law Equation (11) for the controlling torques M_i. Taking into account the condition ω(0)=ω(T)=0, optimal solution for our problem Equations (1)-(6) satisfies the following relations:

$$\varphi_i=f(t)p_i \quad (13)$$

$$J_i\omega_i=a(t)p_i \quad (14)$$

where $a(t)$, $f(t)$ are the scalar functions of time ($a(t) \ge 0$ within the entire interval of time $t\in[0, T]$).

Successive substitution of dependences Equation (13) into Equation (9), by taking into account the relations Equation (14) and $r_i=r_0p_i$, confirms the validity of the found solution, i.e. the solution Equations (13) and (14) for the system of differential Equations (1), (9), (11) and (12) is indeed true (the Equation (14) follow directly from system Equations (1), (11), (12) and (13)). From Equations (7), (9) and (11), we see that the optimal functions $a(t)$, $f(t)$ satisfy the dependence

$$\dot{f}(t)=2a(t)-r_0 \quad (15)$$

Taking into account the Equations (12), (13) and (14), we calculate the left sides and the right sides of Equations (9). The left sides of Equations (9) is identically equal to the right sides if the functions $a(t)$, $f(t)$ satisfy the dependence Equation (15). Control functions M_i are proportional the components p_i of the vector p.

$$M_i=m(t)p_i \quad (16)$$

The function $f(t)$ and constant k_0u_0 determine the scalar function $m(t)$. Optimal function $m(t)$ is:

$$m(t)=\begin{cases} f(t)/(2k_0), \text{ if } f^2(t)\left(p_1^2/J_1+p_2^2/J_2+p_3^2/J_3\right)\le 4k_0^2u_0^2 \\ \dfrac{u_0 \operatorname{sign} f(t)}{\sqrt{p_1^2/J_1+p_2^2/J_2+p_3^2/J_3}}, \text{ if } f^2(t)\left(p_1^2/J_1+p_2^2/J_2+p_3^2/J_3\right)> 4k_0^2u_0^2 \end{cases}$$

Optimal motion corresponding to the Equations (12) and (14) has the unique property $p_1^2/J_1+p_2^2/J_2+p_3^2/J_3=\text{const}$. To check, the given equality is differentiated in time, taking into account Equations (12) and (14), as a result of which we make sure that the resulting derivative is equal to zero after substituting $\dot{p}_i$ according to the Equation (12), and then calculating ω_i using the expressions Equation (14).

Taking into account the laws Equations (11) and (13) and the features of the function $f(t)$ we have

$$m(t)=\begin{cases} f(t)/(2k_0), \text{ if } |f(t)|\le 2k_0m_0 \\ m_0 \operatorname{sign} f(t), \text{ if } |f(t)|>2k_0m_0 \end{cases}$$

where $m_0=u_0/C$; $C=\sqrt{p_{10}^2/J_1+p_{20}^2/J_2+p_{30}^2/J_3}$, $p_{i0}=p_i(0)$

Scalar function $m(t)$ does not overstep the bounds of the range from $-m_0$ till m_0, therefore $|M|\le m_0$.

2.4 Main Properties and Admissible Versions of Optimal Control

A formulated problem of optimal control Equations (1)–(6) is solved completely. For all type of optimal control $a(0)>0$, $a(T)<0$, and the functions $a(t)$ and $m(t)$ are related by the expressions: $a(t)=\int_0^t m(t)dt$, $a(0)=0$, $a(T)=0$, $\dot{a}=m(t)$, (the connection $\dot{a}=m(t)$ is obtained from (1) and (11), (12)). For optimal function $f(t)$, we have the following differential equation: $\ddot{f}=2\dot{a}=2m(t)$. If $|f(t)|\le 2k_0m_0$, then $\ddot{f}=f/k_0$; if $|f(t)|>2k_0m_0$, then $\ddot{f}=2m_0\operatorname{sign}f(t)$. Note that $f(t)$ is smooth function of time during the entire period of time $t\in[0, T]$. Figure 1 demonstrate the behavior of the functions $m(t)$ and $a(t)$ under optimal control, where t_1 and t_2 are the times of occurrence of the equalities $f(t)=2k_0m_0$ and $f(t)=-2k_0m_0$. Optimal function $a(t)$ are described by the following dependence:

$$a(t)=\begin{cases} m_0t, \text{ if } t\le t_1 \\ (r_0+C_2\,exp\left((t-t_1)/\sqrt{k_0}\right)/\sqrt{k_0}-C_1\,exp\left((t_1-t)/\sqrt{k_0}\right)/\sqrt{k_0})/2, \quad (17) \\ \text{if } t_1<t<t_2,\ m_0(T-t), \text{ if } t\ge t_2 \end{cases}$$

where C_1 and C_2 are some constants. Accordingly, the function $m(t)$ has the form:

$$m_0, \text{ if } t\le t_1$$

$$[C_1\,exp\left((t_1-t)/\sqrt{k_0}\right)+C_2\,exp\left((t-t_1)/\sqrt{k_0}\right)]/(2k_0), \text{ if } t_1<t<t_2 \quad (18)$$

$-m_0$, if $t \geq t_2$

because the function $f(t)$ has the following form $f(t) = C_1\, exp\,((t_1 - t)/\sqrt{k_0}) + C_2\, exp\,((t - t_1)/\sqrt{k_0})$, during time interval when $|f(t)| \leq 2k_0m_0$.

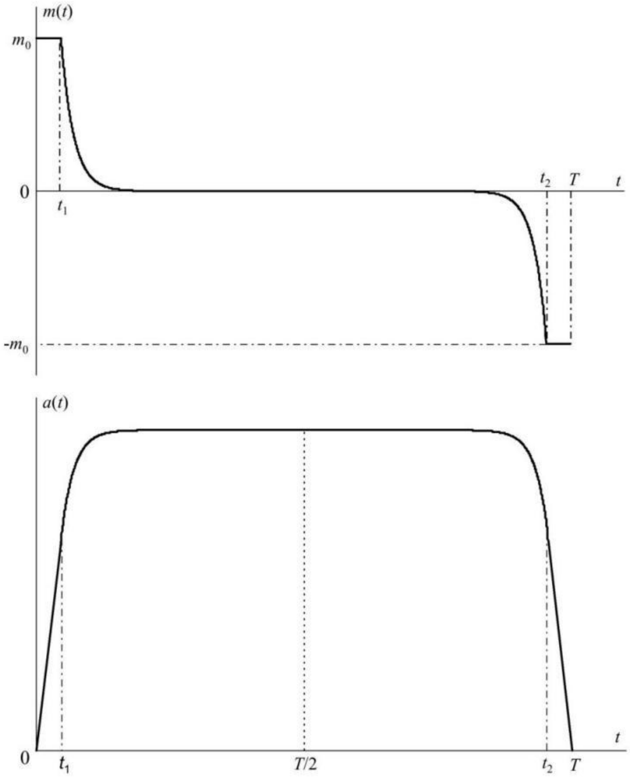

Figure 1. Typical form of optimal functions *m*(*t*) and *a*(*t*).

Notice that the values C_1 and C_2 should satisfy the equalities $C_1\, exp\,((2t_1 - T)/\sqrt{k_0}) + C_2\, exp\,((T - 2t_1)/\sqrt{k_0}) = -2k_0m_0$; $C_1 + C_2 = 2k_0m_0$, since the function $f(t)$ is equal $f(t_1)=2k_0m_0$ at the instant t_1, but the function $f(t)$ will equal $f(t_2)=-2k_0m_0$ at the instant of time t_2. From here we obtain the following relations:

$$C_1 = -\, C_2\, exp\,((T - 2t_1)/\sqrt{k_0}),$$

$$C_1 = 2k_0m_0/(1 - exp\,((2t_1 - T)/\sqrt{k_0})),$$

$$C_2 = 2k_0m_0/(1 - exp\,((T - 2t_1)/\sqrt{k_0})),$$

because $a(t_2)=a(t_1)=m_0t_1$ и $t_2=T-t_1$. We easily see that $f(T)=-f(0)$ and $f(T/2)=0$, $\dot{f}(T/2) < 0$.

The boundary-value problem of the maximum principle is to specify the vector p(0) and the value $r_0>0$, under which the solution of the system of Equations (1), (2), (9), (11) and (12) with initial conditions (4) and relation $r_i=r_0p_i$ satisfies the boundary conditions (5).

The presence of the switching points t_1, t_2 depends on the integral

$$Q = \int_0^T a(t)dt \quad (19)$$

which does not depend on character of changing the function $a(t)$ and it is determined exclusively by the quaternions Λ_{in}, Λ_f and the moments of inertia J_1, J_2, J_3[14] (the value Q is calculated simultaneously with the vector p_0).

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If $m_0T^2<4Q$, then solution of the problem (1)-(6) is absent, since $T_{fast}=2\sqrt{Q/m_0}$ is the minimum possible duration of a turn of solid body with moments of inertia J_1, J_2, J_3 from the position (4) into the position (5) under the constraint (3)[9].

If $m_0T^2=4Q$, then |M|=const=m_0 during the entire interval of control $t\in[0, T]$ independently from the coefficient k_0.

If $Q(\exp(T/\sqrt{k_0})-1)\le 2m_0[T(\exp(T/\sqrt{k_0})+1)\sqrt{k_0}/2-k_0(\exp(T/\sqrt{k_0})-1)]$, then in any time |M|≠const, because $f(0)\le 2k_0m_0$ in this case, and $|f(t)|\le 2k_0m_0$ within period of time $t\in[0, T]$. The restriction (3) is insignificant for this combination of values Q, k_0, m_0 and T.

If $Q(\exp(T/\sqrt{k_0})-1)>2m_0[T(\exp(T/\sqrt{k_0})+1)\sqrt{k_0}/2-k_0(\exp(T/\sqrt{k_0})-1)]$, then $m(0)=m_0$ and time intervals with |M|=const are inevitable (they necessarily are present in optimal control). For period when |M|≠const, we have the equation $\ddot{f}=f/k_0$, and the functions $a(t)$, $f(t)$ are $a(t)=m_0t_1+(C_2\exp((t-t_1)/\sqrt{k_0})-C_1\exp((t_1-t)/\sqrt{k_0})+C_1-C_2)/(2\sqrt{k_0})$, $f(t)=C_1\exp((t_1-t)/\sqrt{k_0})+C_2\exp((t-t_1)/\sqrt{k_0})$.

The value of the integral (19) is $Q=m_0t_1(T-t_1)+\frac{\sqrt{k_0}}{2}\int_0^{T-2t_1}(C_2\exp(\tau/\sqrt{k_0})-C_1\exp(-\tau/\sqrt{k_0})+C_1-C_2)d\tau$, $C_2=-C_1\exp((2t_1-T)/\sqrt{k_0})$.

The switching point t_1 is determined using the following equation

$$\frac{(Q-m_0t_1(T-t_1))(\exp((T-2t_1)/\sqrt{k_0})-1)}{(T-2t_1)(\exp((T-2t_1)/\sqrt{k_0})+1)\sqrt{k_0}/2-k_0(\exp((T-2t_1)/\sqrt{k_0})-1)}=2m_0$$

The value $t_2=T-t_1$. The constants C_1 and C_2, and r_0 are calculated using the formulas

$$C_1=\frac{k_0(Q-m_0(T-t_1)t_1)}{0.5\sqrt{k_0}(T-2t_1)(\exp((2t_1-T)/\sqrt{k_0})+1)+k_0(\exp((2t_1-T)/\sqrt{k_0})-1)}$$

$$r_0=2m_0t_1+\frac{C_1-C_2}{\sqrt{k_0}} \quad (20)$$

$$C_2=\frac{k_0(m_0(T-t_1)t_1-Q)}{0.5\sqrt{k_0}(T-2t_1)(\exp((T-2t_1)/\sqrt{k_0})+1)-k_0(\exp((T-2t_1)/\sqrt{k_0})-1)}$$

Maximal magnitude of angular momentum L_{max} is equal

$$L_{max}+m_0t_1[C_2\exp((T/2-t_1)/\sqrt{k_0})-C_1\exp((t_1-T/2)/\sqrt{k_0})+C_1-C_2]\sqrt{k_0}/2_{max=m_0t}$$

Key feature of optimal control is a constancy of proportion between kinetic energy of rotation E_k and the squared modulus of the angular momentum of a spacecraft.

$E_k=a^2(p_1^2/J_1+p_2^2/J_2+p_3^2/J_3)/2$ $E_k/|L|^2=(p_1^2/J_1+p_2^2/J_2+p_3^2/J_3)/2=\text{const}$, because $a^2=|L|^2$ (it is shown from (13)) and $p_1^2/J_1+p_2^2/J_2+p_3^2/J_3=\text{const}=p_{10}^2/J_1+p_{20}^2/J_2+p_{30}^2/J_3$, where p_{10}, p_{20}, p_{30} are the components of the vector $p_0=p(0)$.

The Hamiltonian H does not depend on time in explicit form, therefore H = const along optimal trajectory within the entire interval of control $t \in [0, T]$[29]. At initial instant t=0, we have $H(0) = f(0)m_0C^2 - k_0u_0^2$; at final instant t=T, we have $H(T) =- f(T)m_0C^2 - k_0u_0^2$. Whence $f(T)$=-$f(0)$ and $f(0) = (H(0) + k_0u_0^2)/(u_0C)$. At the instant of time t=T/2 we have $f(T/2)$=0 and $H(T/2)$=$L_{max}C^2(r_0- L_{max})$. At the time t=t_1, we have $H(t_1) = u_0^2(k_0 - t_1^2) + r_0t_1u_0C$ =$H(t_2)$. During the segments when $|M|$=const, the functions $a(t)$, $f(t)$ are:

$f(t)$=$H/(u_0C)+k_0m_0- r_0t+m_0t^2$, if $t \le t_1$;

$f(t)$=$r_0(T- t)- m_0(T- t)^2- H/(u_0C)- k_0m_0$, if $t \ge t_2$;

$a(t)$=m_0t, if $t \le t_1$; and $a(t)$=$m_0(T- t)$, if $t \ge t_2$.

We see that $H = u_0^2(k_0 + t_1^2) + (C_1 - C_2)u_0t_1C/\sqrt{k_0}$, for any time $t \in [0,T]$. Accordingly, $f(0)$=$m_0(2k_0 + t_1^2) + t_1(C_1 - C_2)/\sqrt{k_0}$.

The conditions C_1>0, C_2<0 are satisfied for any values Q, k_0 and T; hence, r_0>0 and $a(0)$>0, $a(T)$<0 are ensured. Thus, in the time interval $t_1<t<t_2$ when $|f(t)|<2k_0m_0$, optimal function $a(t)$ is $a(t) = m_0t_1 + (C_2\, exp\,((t - t_1)/\sqrt{k_0}) - C_1\, exp\,((t_1 - t)/\sqrt{k_0}) + C_1 - C_2)/(2\sqrt{k_0})$ If $f(t) \ge 2k_0m_0$, then $a(t)$=m_0t; if $f(t) \le -2k_0m_0$, then $a(t)$= $m_0(T–t)$.

About the function $f(t)$, we can write:
(a) within the segment $t<t_1$ when $f(t)>2k_0m_0$, optimal function $f(t)$ is calculated by the expression:

$$f(t) = m_0(2k_0 + t_1^2 + t^2) + (t_1 - t)(C_1 - C_2)/\sqrt{k_0} - 2m_0t_1t$$

(b) within the segment $t_1 \le t \le t_2$ when $|f(t)| \le 2k_0m_0$, optimal function $f(t)$ has the form:

$$f(t) = C_1\, exp\,((t_1 - t)/\sqrt{k_0}) + C_2\, exp\,((t - t_1)/\sqrt{k_0}),\ m(t)=f(t)/(2k_0)$$

(c) within the segment $t>t_2$ when $f(t)<-2k_0m_0$, optimal function $f(t)$ is calculated with use of the formula

$$f(t) = 2m_0t_1(T - t) - m_0(2k_0 + (T - t)^2 + t_1^2) - (t_1 - T + t)(C_1 - C_2)/\sqrt{k_0}$$

Solution of the optimal turn problem is described by the Equations (12-14); control functions M_i and angular velocities ω_i change according to Equations (14) and (16). The vector p_0 and integral Q are found after solving two-point boundary-value problem. The characteristics Q, m_0 completely determine the program of spacecraft rotation. The programmed torque M and the quaternion Λ are related as follows:

$$M = m(t)\tilde{\Lambda} \circ \Lambda_{in} \circ p_0 \circ \tilde{\Lambda}_{in} \circ \Lambda$$

moreover $m(t)$ changes according to (18).

Optimal rotation has the property of symmetry (for the functions a (t) and $m(t)$ also), and we have the following dependences:

$m(0) =- m(T) > 0,\ a(t) \ge 0,\ m(T - t) =- m(t),\ a(T–t)=a(t)$

$\int_0^{T/2} |m(t)|dt = \int_{T/2}^{T} |m(t)|dt,\ \int_0^{T/2} a(t)dt = \int_{T/2}^{T} a(t)dt$

$\boldsymbol{\Lambda} \circ \boldsymbol{M}(\boldsymbol{T - t}) \circ \tilde{\boldsymbol{\Lambda}} =- \boldsymbol{\Lambda} \circ \boldsymbol{M}(\boldsymbol{t}) \circ \tilde{\boldsymbol{\Lambda}}$, $\Lambda \circ \mathbf{L}(T-t) \circ \tilde{\Lambda} = \Lambda \circ \mathbf{L}(t) \circ \tilde{\Lambda}$ $\max_{t<T/2} m(t) = -\min_{t>T/2} m(t) = m(0)$, $L_{max} = \max_{0<t<T} \sqrt{J_1^2\omega_1^2 + J_2^2\omega_2^2 + J_3^2\omega_3^2} = |\boldsymbol{L}(T/2)|$

As we can see in (14), p is the ort of angular momentum L. Optimal functions $\omega_i(t)$, $\varphi_i(t)$, $p_i(t)$ satisfy the requirements (13), (14) in which $p_i(t)$ is solution of system (12). The expression (16) determine optimal control; the vectors M and L are collinear at any time $t\in[0, T]$, direction of angular momentum L is constant in inertial coordinate system. The control (16) is indeed optimum because it is unique solution to the system (1), (7), (9), (11).

For unlimited control, optimal solution is simplified:

$$\mathrm{M}=f(t)\mathrm{p}/(2k_0),\ f(t)=C_1\,exp\,(-t/\sqrt{k_0})+C_2\,exp\,(t/\sqrt{k_0}),\ C_1=-C_2\,exp\,(T/\sqrt{k_0})$$

$$a(t)=\sqrt{k_0}\,[C_2\,exp\,(t/\sqrt{k_0})-C_1\,exp\,(-t/\sqrt{k_0})+C_1-C_2]\big/2,\ r_0=(C_1-C_2)/\sqrt{k_0}$$

Where $C_1=k_0Q\big/[k_0(\,exp\,(-T/\sqrt{k_0})-1)+T\sqrt{k_0}(\,exp\,(-T/\sqrt{k_0})+1)/2\,]$

$$C_2=k_0\,Q\big/[k_0(\,exp\,(T/\sqrt{k_0})-1)-T\sqrt{k_0}(\,exp\,(T/\sqrt{k_0})+1)/2\,]$$

For control bounded by the requirement (3), optimal solution is $\mathrm{M}=m(t)\mathrm{p}$, where $m(t)=u_0/C$, if $f(t)>2k_0m_0$; $m(t)=f(t)/(2k_0)$, if $|f(t)|\le 2k_0m_0$; and $m(t)=-u_0/C$, if $f(t)<-2k_0m_0$.

The characteristic Q, and optimal values m_0, p_0 are determined exclusively by the values u_0, Λ_{in}, Λ_f and J_1, J_2, J_3, and the times t_1, t_2 depend on k_0. At the instant of time $t=T/2$, the control torque M changes its direction to the opposite one, the modulus |L| is maximum ($|\mathrm{L}(T/2)|=L_{\max}$).

The desired value t_1 is found within the interval $t_1\in[0, t_0]$, where $t_0=(T-\sqrt{T^2-4Q/m_0})/2$. If we suppose $t_1=t_0$, then it means $C_1=C_2=0$ (it follows from (20)) and a=const, $f(t)=0$, but such functions can not be in optimal control, since optimal torque $\mathrm{M}\neq 0$ in segment $t\in[t_1, t_2]$, and optimal function $f(t)$ should change from $2k_0m_0$ to the level $-2k_0m_0$. Consequently, $t_1<t_0$ for optimal motion.

2.5 Special Cases of Optimal Reorientation

Control functions are formed by the laws (16), (18), for which it is necessary to know p_1, p_2, p_3 at each current time t. An analytical solution of system (2), (12), (14) exists only for dynamically symmetric and spherically symmetric bodies. ~~In the case~~ of a spherically symmetric spacecraft ($J_1=J_2=J_3$), the solution is: $p_i(t)=\mathrm{const}=p_{i0}=\nu_i/\sqrt{\nu_1^2+\nu_2^2+\nu_3^2}$, $M_i(t)=m(t)p_{i0}$, $\omega_i(t)=a(t)p_{i0}/J_i$, where time functions $a(t)$ and $m(t)$ are specified by the parameters $m_0=u_0\sqrt{J_1}$, $Q=2J_1\arccos\nu_0$ and k_0 (see Section 2.4); $\nu_0, \nu_1, \nu_2, \nu_3$ are elements of quaternion $\Lambda_t=\widetilde{\Lambda}_{in}\circ\Lambda_f$. The trajectory of rotation $\Lambda(t)$ has analytical form $\Lambda(t)=\Lambda_{in}\circ e^{p_0 s(t)/(2J_1)}$, $s(t)=\int_0^t a(t)dt$.

For dynamically symmetric spacecraft ($J_2=J_3$), problem of optimal control (1)-(6) can be solved completely also (the axis OX is considered to be the axis of symmetry for the sake of concreteness). With such a distribution of masses, $p_1=\mathrm{const}=p_{10}$ and spacecraft optimal motion is the simultaneous rotation around the vector p, fixed in the inertial coordinate system, and around the longitudinal axis OX. The angle ϑ between angular momentum L and the longitudinal axis is constant, and modulus of angular momentum is proportional to a velocity around the longitudinal axis. The angular velocities around p and the axis OX change proportionally with a constant coefficient of proportionality, due to which we have[9,12]

$$\Lambda_f=\Lambda_{in}\circ e^{p_0\beta/2}\circ e^{e_1\alpha/2},\ \alpha=p_{10}\beta\left(\frac{J_2}{J_1}-1\right)\ (21)$$

where $\mathbf{e}_1$ is the unit vector of the symmetry axis OX; α, β are the angles of spacecraft rotation around the OX axis and around p ($|\alpha|\le\pi$, $0\le\beta\le\pi$). Solution p(t) has analytical form[9,12]:

$p_1=p_{10}=\cos\vartheta$, $p_2=p_{20}\cos\kappa+p_{30}\sin\kappa$, $p_3=-p_{20}\sin\kappa+p_{30}\cos\kappa$ (22)

$$\kappa = \frac{J - J_1}{J} \int_0^t \omega_1(t) dt$$

where $p_{i0}=p_i(0)$; $J=J_2=J_3$; the longitudinal velocity $\omega_1(t)$ is determined from (14) given that p_1=const=p_{10}. The values α, β and p_{i0} are found by quaternions Λ_{in} and Λ_f using the Equation (21). Notice, optimal value p_0 corresponding to optimal solution can be defined with use of device[33].

In this particular case, the described solution differs from the known paper[12], since all control variables $M_i(t)$ are the continuous functions of time. Angular velocities ω_i are calculated by the Equations (14) and (22). Optimal functions $a(t)$ and $m(t)$ are determined by the programs (17), (18) and depend on the values T, m_0, Q, which are computed definitely by given values Λ_{in}, Λ_f, u_0, k_0 and J_1, J_2, J_3. We write the sought optimal controls $M_i(t)$ in analytical form:

$M_1=m(t)p_{10}$, $M_2=m(t)\sqrt{1-p_{10}^2}\,sin\,(\kappa+\gamma)$, $M_3=m(t)\sqrt{1-p_{10}^2}\,cos\,(\kappa+\gamma)$,

where $\gamma = arcsin\left(p_{20}\middle/\sqrt{1-p_{10}^2}\right)$,

if $p_{30} \geq 0$,or $\gamma = \pi - arcsin\left(p_{20}\middle/\sqrt{1-p_{10}^2}\right)$,

if $p_{30} < 0$ ($|p_{10}| \neq 1$); the case $|p_{10}|=1$ means flat rotation around the OX axis, therefore it is not considered.

Optimal trajectory of dynamically symmetric spacecraft $\Lambda(t)$ is represented in analytical form $\Lambda(t) = \Lambda_{in} \circ e^{p_0\sigma/2} \circ e^{\mu e_1/2}$, where $\sigma = J_2^{-1}\int_0^t a(t)dt$; $\mu = p_{10}\sigma(J_2 - J_1)/J_1$.

The parameters p_0, m_0, T for dynamically symmetric body are found more simply (calculation of (19) is simplified also); $Q=J_2\beta$, since $|L| = J_2\dot{\beta}$, where $\dot{\beta}$ is rotation speed around angular momentum L (we specify $\beta\geq0$, $\dot{\beta}\geq0$). The values L_{max}, G depend on β. For (6) to be minimal value, it is necessary that the angle β be the minimum possible, for which $\beta\leq\pi$ (the Equation (21) is supplemented by the requirement $0\leq\beta\leq\pi$). We can prove that solution to system (21) exists for any Λ_{in}, Λ_f and any J_1, $J_2=J_3$.

For an asymmetric spacecraft (when $J_1\neq J_2\neq J_3$), we can solve the system (2), (12), (14) by numeric methods only (e.g., using a method of successive approximations or iterations methods with consecutive approach to true solution). One of such methods was described in detail in the known paper[9]. We know that the solution p(0) which satisfies the conditions $\Lambda(0)=\Lambda_{in}$, $\Lambda(T)=\Lambda_f$ and second equality of (21) for the system of Equations (2), (12) and (14) does not depend on a type of changing the magnitude of angular momentum[14] (therefore, we take a=const≠0 in (14) for search of p(0)). The behavior of p(t) and solution of Equation (14) corresponds to rotation by inertia of solid body when $a(t)$=const. To compute the desired vector p(0), we must solve the boundary-value problem $\Lambda(0)=\Lambda_{in}$, $\Lambda(T)=\Lambda_f$, taking into account the Equations (1) and (2) in which M_i=0. This boundary-value problem of a turn can be solved by the iteration method (see the method[34] and the system[35]). As a result, angular velocity vector at initial instant of time ω_{cal}, for which a spacecraft rotates with its free motion (M=0) from the state $\Lambda(0)=\Lambda_{in}$, $\omega(0)=\omega_{cal}$ into the state $\Lambda(T)=\Lambda_f$, will be found. The vector p_0=p(0) relates to ω_{cal} as follows:

$$p_{i\,0} = \frac{J_i\omega_{i\ \mathrm{cal}}}{\sqrt{J_1^2\omega_{1\mathrm{cal}}^2 + J_2^2\omega_{2\mathrm{cal}}^2 + J_3^2\omega_{3\mathrm{cal}}^2}}$$

Other computation schemes are useful in some specific cases[36].

2.6 Numerical Example and Data of Mathematical Simulation

In this section, we give numerical solution of optimal turn problem with minimal value of (6). For example, we consider 150° rotation into position that is characterized by the quaternion Λ_f with elements λ_0=0.258819; λ_1=0.683013; λ_2=0.258819; λ_3=0.591505. Direction of body-fixed axes and the axes of inertial basis coincide in initial state, and $\omega(0)=\omega(T)=0$ also. We construct optimal control program for spacecraft's reorientation from a state $\Lambda(0) = \Lambda_{\text{in}}$, $\omega(0)=0$ into a state $\Lambda(\text{T}) = \Lambda_{\text{f}}$, $\omega(T)=0$ within T=300s. Numerical solution of controlled turn problem in formulation (1)–(6) is presented for case when u_0=0.044Nkg$^{-1/2}$ and k_0=10s^2, and mass-inertial characteristics of a spacecraft are as follows: J_1=4710kg·m^2, J_2=17160 kg·m^2, J_3=18125 kg·m^2.

To solve the two-point boundary-value problem, we assume that $a(t)$=const in (14) (and │L│=const), since a character of behavior of the function $a(t)$ does not influence on the sought value p$_0$[14]. After solving a boundary-value problem of a turn from position $\Lambda(0)=\Lambda_{in}$ into position $\Lambda(T)=\Lambda_f$, the calculated vector p$_0$ and (19) were found p$_0$={0.381804; 0.1941395; 0.9036236}, Q=28907.5Nms^2. Accordingly, m_0=5Nm. We see that $m_0T^2>4Q$, hence the given turn can be implemented. Also, $Q(exp\,(T/\sqrt{k_0})-1)>2m_0[T(exp\,(T/\sqrt{k_0})+1)\sqrt{k_0}/2-k_0(exp\,(T/\sqrt{k_0})-1)]$ for k_0 and m_0, therefore │M│=m_0 at initial and final instants of time, optimal rotation includes two interval of time when│M│=const. Optimal process has three interval of changing the control torque M: fist interval is intensive spin-up of a spacecraft (when $m(t)$=const=m_0), second phase is control with exponential change of $m(t)$ (the function $m(t)$ decreases from $m(t)=m_0$ to $m(t)=-m_0$), and finally, intense braking at the end of a turn (during this stage $m(t)$=const=$-m_0$). The switching points are t_1=17.576s and t_2=282.424s, constant r_0=178.92Nms. The angular momentum reaches the maximum value $L_{\max}$=103.69Nms at the time t=150s.

Results of numeric simulation of optimal turn and graphical illustration of spacecraft motion are presented on Figures 2-4. The graphs of changing the angular velocities are shown in Figure 2 (angular velocities ω_i given in degree/s). The velocity ω_1 is of constant sign, and the nature of its change repeats the behavior of the angular momentum modulus (in contrast to ω_2 and ω_3) due to the OX is the longitudinal axis of the spacecraft. Figure 3 shows variation of quaternion elements for $\Lambda(t)$ during slew maneuver ($\lambda_0(t)$, $\lambda_1(t)$, $\lambda_2(t)$, $\lambda_3(t)$ reflect the current attitude of a spacecraft). Finally, Figure 4 demonstrates a dynamics of variables $p_1(t)$, $p_2(t)$, $p_3(t)$ which form unit vector p. We see that variation of the projection p_1 is a lot less than variation of the projections p_2 and p_3. The variables p_i, ω_i and λ_j are smooth functions of time (unlike control functions M_i). Optimal function $m(t)$ is presented in Figure 5 for the considered turn ($m(t)$ given in Nm). Figure 6 illustrates the change in energy of rotation during optimal turn of a spacecraft (energy E_k given in J). Rotation energy during maneuver less than $E_{\max}$=0.42J.

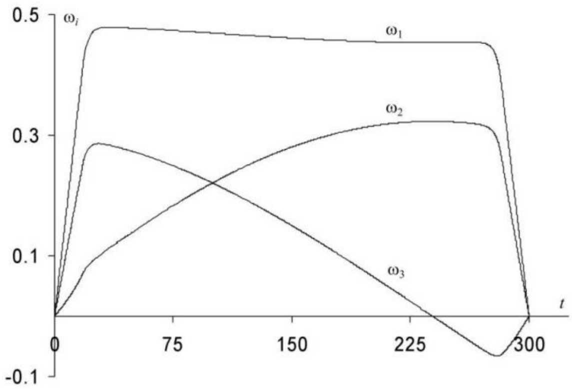

Figure 2. Changing the angular velocities of a spacecraft during optimal turn.

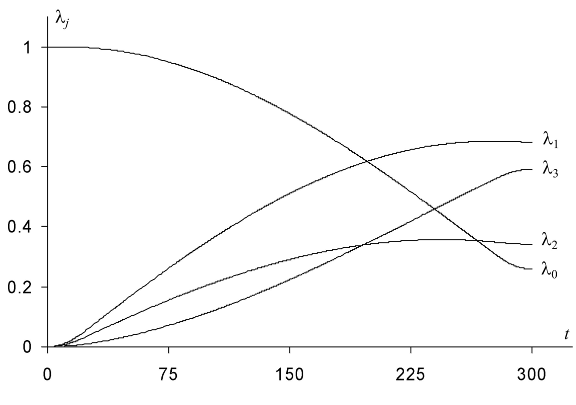

Figure 3. Elements of quaternion Λ(*t*) during spatial turn.

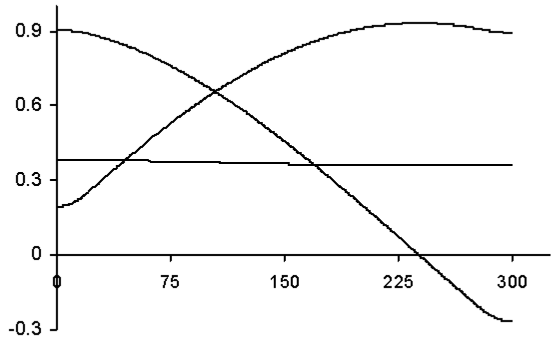

Figure 4. The functions $p_1(t)$, $p_2(t)$, $p_3(t)$ during optimal maneuver.

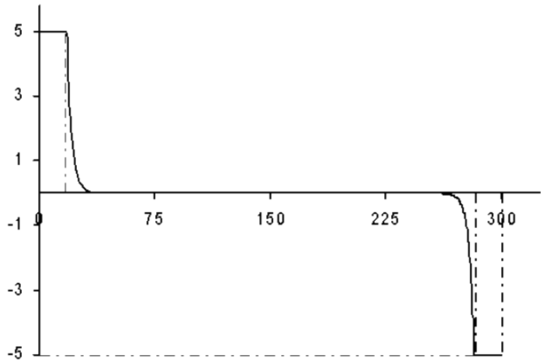

Figure 5. Optimal function m(t) for model turn.

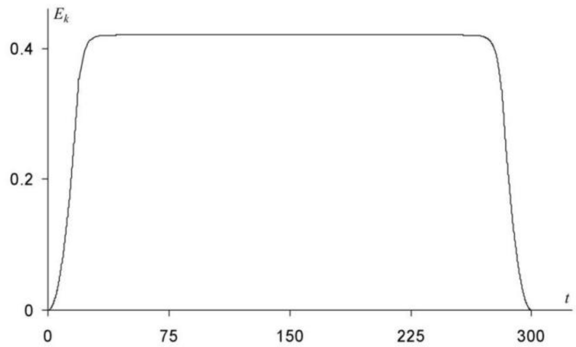

Figure 6. The change in energy of rotation Ek in a turn process.

3 RESULTS AND DISCUSSION

The optimal control problem for a turn of solid body (in particular, spacecraft) from initial position into the prescribed position is investigated in detail. For optimization, a chosen indicator of quality combines the contribution of the control forces (in sense of energy costs) expended for implementing the maneuver of a turn and the integral of the kinetic energy of rotation in a given proportion. We found key properties of optimal turn and a type of trajectory that corresponds to the criterion (6). It is proved that proportion between the squared modulus of angular momentum and kinetic energy of spacecraft rotation is constant during the maneuver.

Quaternion models and the Pontryagin's maximum principle were used for solving the proposed problem of control. We have written out the Hamilton-Pontryagin function and the conjugate system of equations for the formulated optimization problem. Also, analytical expressions for optimal control functions were obtained. The construction of optimal control is determined on the base of the necessary conditions of optimality which were formed in analytic form. Relations for determination of spatial motion of a spacecraft were given; the uniqueness of optimal solution is confirmed.

Obtained control method differs from all known decisions. Main difference is use of new functional of quality. The presence of integral of the kinetic energy reduces maximal rotation energy. The coefficient k_0, specifying the proportion between costs of control resources and integral of rotation energy, determines how gentle the change in the angular momentum modulus will be during the optimal turn. Other principal difference consists that presence of segments of rotation with the constant modulus of control torque is not excluded due to narrow-mindedness of control during optimal turn. In our work, all possible variants of optimal control are determined and described in detail, the condition (criterion) allowing to specify a type of optimal control basing on the given coefficient k_0 in the minimized functional and domain of permissible control. Depending on value k_0, one of two versions of control can be optimum: (a) rotation with changing modulus of control torque during entire maneuver; (b) the control with segments of rotation with constant maximum modulus of control torque entitled as an acceleration of rotation in a beginning of a turn, and braking at the end of a turn. We have given analytical equation which establishes a relation between duration of segments of rotation with constant modulus of control torque and the coefficient k_0 in the minimized index, and the maximum possible magnitude of control torque m_0. The formulas for calculating a maximal energy of rotation and maximum modulus of angular momentum are written in analytical form. The procedure for implementing the optimal control mode is described.

The obtained results differ from solution in the paper[12], where relay control was found in contrast to continuous control that is optimal in our problem. Computational expressions for calculating the basic characteristics of slew maneuver are presented. Example and the results of numerical simulation for spacecraft rotation under optimal control are given, demonstrating the behavior of motion parameters. In particular case, for a dynamically symmetric spacecraft, complete solution of the problem of a spatial turn is presented: a system of equations is obtained in analytical form, from which the solution of the boundary-value problem is directly found and the necessary constants of the control law are calculated (we can use the known device[33]).

In recent years, due to the increase in the duration of the active existence of the spacecrafts (more than 10-15 years) and an applying of precise attitude control systems, interest in ERJ engines has increased significantly[37]. The indisputable advantages of ERJ engines are the possibility of a small value of a single thrust pulse and accurate pulse dosing, which ensures especially precise orientation. Due to the unimaginably high values of the specific pulse (up to 6000s), a wide use of ERJ engines in spacecraft systems (including for orienting a spacecraft) is one of the leading and natural trends in space activities in the world. At the present time, many foreign spacecrafts use ion thrusters for attitude control (for example, XIPS-25 ion thrusters developed by Boeing Space Systems were used to control spacecraft's orientation in the US space program). In this case, the consumed electrical energy very close is estimated by the value proportional to the first term in the index (6); the second term in (6) limits the kinetic energy of rotation, making it as little as possible for the given turning time under the limited control torque, which is also highly desirable in space flight. Taking into account the need for an all-round reduction in the power consumption of the ERJ engine for controlling the spacecraft, together with a desired decreasing of rotation energy, a choice of the minimized functional in the form (6) becomes clear.

The issues of cost-effectiveness of spacecraft's motion control remain relevant; thus, the problem solved in this article is important in practice. The proposed solution is different from all known ones; during the optimal turn of a spacecraft, both the control functions and the phase variables are smooth continuous functions of time.

4 CONCLUSION

Optimal control program of spacecraft's reorientation with minimal costs of energy has been found; it is demonstrated that the control when angular momentum is parallel to a controlling torque within the entire interval of spatial turn is optimum. The issues of profitability and economical control of spacecraft rotations is relevant in present time, therefore the studied problem is very important. The solved problem differs from other problems with a combined functional in index form, in the presence of restrictions on the control and does not concern to an axi-symmetric body[21-23]. The obtained results demonstrate that the designed control method of spacecraft's three-dimensional reorientation is feasible in practice.

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Conflicts of Interest

The author declared that there are no conflict of interest.

Author Contribution

Levskii M was responsible for the original draft, including methodology, validation, mathematical formulating and analysis of formulas, writing, conceptualization, supervision, numerical modeling and design of the figures that illustrated example of mathematical simulation of optimal rotation.

Abbreviation List

ERJ, Electric-rocket engines

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