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Research Article

How Shocks Affect Markets: A Novel Dynamical Macroeconomic Model to Explain Adjustment of Markets and Equilibria

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Abstract

Objective: Most macroeconomic models are based on representative agents with identical preferences for all consumers and production technology for all producers, i.e., an assumption too simplistic if not unrealistic to model the real world. Similarly, the models revolve around a general equilibrium for all markets which seldom exists in a dynamic and rapidly changing and evolving world where shocks keep happening too frequently to imagine all markets to stay put in an economy. There is a lack of robustness of macroeconomic models with respect to inflexible assumptions they are based on (including but not limited to specific structural forms for utility functions and production technology). This paper provides a foundation stone for a more realistic macroeconomic modeling based on practical behavior of economic agents with minimum number of assumptions without use of specific and complex structural forms as compared to those in the existing literature.

Methods: This paper captures an interaction of three markets, i.e., goods, labor, and capital. Dynamic optimization problems of agents in all three markets have been solved to find expressions regarding their individual decisions, which have been solved simultaneously to get a nonhomogeneous linear system of differential equations, for which conditions for a unique solution has been specified. Also, conditions regarding stability and existence of an equilibrium have been stipulated.

Results: It provides results which are robust to heterogeneous consumers and producers exhibiting bounded rationality. It models macroeconomy based on easily measurable empirical components. After estimating and substituting empirical parameter values in the system of differential equations and solving them, the response of three markets can be predicted. The model captures not only both initial and final sets of equilibria before and after shocks to all markets, rather it predicts the full adjustment path of all markets from initial to final equilibriums after various kinds of shocks happen to one or more markets.

Conclusions: Optimal policies, such as monetary policy, taxation, inflation control, employment, trade, remittances, etc., affecting one or more of the three markets subject to relevant constraints can be derived based on a system of differential equations. The methodology employed for three markets can be extended to n number of markets in an economy.

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Keywords: macroeconomic modeling, heterogeneous agents, bounded rationality, equilibrium, disequilibrium, coordination

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1 INTRODUCTION

Four important critiques of dynamic stochastic general equilibrium (DSGE) models from an agent-based computational economics (ACE) perspective revolve around heterogeneity, disequilibrium, complexity, and rationality. Modern DSGE models often answer one or more of these critiques. Although it is difficult to strictly classify and compare ACE models, it is possible to identify four major families of models within the macro-ACE literature. Some New Keynesian models incorporate insights of ACE models into DSGE; however, they do not incorporate local interaction and disequilibrium.

Most of macroeconomic models are based on representative agents with identical preferences for all consumers and production technology for all producers, i.e., an assumption too simplistic if not unrealistic to model the real world. Similarly, the models revolve around a general equilibrium for all markets which seldom exists in a dynamic and rapidly changing and evolving world where shocks keep happening too frequently to imagine all markets to stay put in an economy. The learning literature adopts two basic approaches to modelling boundedly rational expectations. The first is usually referred to as statistical learning, where agents are competent econometricians who make observations of the price, have some idea of the data generating process and estimate it using standard techniques. The second approach assumes that agents use simple heuristic forecasting rules. A general formulation that nests examples found in the literature is an adaptive expectations rule.

There is a lack of robustness of macroeconomic models with respect to inflexible assumptions they are based on (including but not limited to specific structural forms for utility functions and production technology). For example, if an assumption of homogeneous agents is relaxed in a DSGE model, it might break down instead of showing robustness with regard to the results it predicts. A model based on practical must be robust with regard to variations in agents' behaviors, especially, until their distribution in an economy stays the same. Secondly, a sound model must be based on measurable parameters to be empirically estimated and used practically with little discretion with practitioners.

This paper provides a foundation stone for a more realistic macroeconomic modeling based on practical behavior of economic agents with minimum number of assumptions without use of specific and complex structural forms as compared to those in the existing literature. It provides results which are robust to heterogeneous consumers and producers exhibiting bounded rationality; and equilibrium expressions as well as disequilibrium paths after shocks happen to various markets. It models macroeconomy based on easily measurable empirical components, such as slopes of supply and demand curves in various markets.

Gatti, Di Guilmi, Gaffeo et al.[1] criticizes reductionist approach of using a representative agent in macroeconomic models ignoring heterogenous preferences and endowments, including non-normal distributions and interactions between agents due to which DSGE models do not allow any room for emergent macroscopic patterns. Howitt's[2] diagnosis is that macroeconomic theory has become distracted by its preoccupation with states of equilibrium, a preoccupation that inhibits analysis of a market economy's coordination mechanisms. Woodford[3] reconsiders familiar results in the theory of monetary and fiscal policy when one allows for departures from the hypothesis of rational expectations. Fagiolo and Roventini[4] presents a critical discussion of the theoretical, empirical and political-economy pitfalls of the DSGE-based approach to policy analysis. They suggest that a more fruitful research avenue should escape the strong theoretical requirements of New Neoclassical Synthesis (NNS) models (e.g., equilibrium, rationality, representative agent, etc.) and consider the economy as a complex evolving system, i.e., as an ecology populated by heterogenous agents, whose far-from-equilibrium interactions continuously change the structure of the system. Rogers[5] argues that new DSGE model is impossible to interpret or to be used as a basis for advice on monetary policy.

Among ACE models, Russo, Catalano, Gaffeo, Gallegati et al.[6] talks about various kinds of macroeconomic models historically used by economists and their empirical performance; and proposes a model for economy which consists of one sector with idiosyncratic R&D shocks at the firm level; firms' pricing strategies are boundedly rational, and evolve according to a specified heuristic. Deissenberg, Van Der Hoog and Dawid[7] describe the general structure of the economic model developed for EURACE and present the Flexible Large-scale Agent Modelling Environment. Gatti, Gallegati, Greenwald et al.[8] model a credit network characterized by credit relationships. Dosi, Fagiolo, Napoletano et al.[9] model a banking sector and a monetary authority setting interest rates and credit lending conditions in a framework combining

Keynesian mechanisms of demand generation, a Schumpeterian innovation-fueled process of growth and Minskian credit dynamics.

There have been models incorporating ACE features into DSGE models. Among those, the complexity framework comprehensively described in Hommes[10], provides a minimal way of generating complex dynamics via heterogeneous agents with varying degrees of rationality. The Brock-Hommes framework has been used by a number of authors to propose a behavioural version of the standard New Keynesian (NK) model with rational expectations (see e.g., Woodford and Walsh[11]). These include Branch and McGough[12], De Grauwe and Kaltwasser[13], Massaro[14], Jang and Sacht[15], and Galí[16]. Branch and McGough[17] provides a recent survey. Dilaver, Calvert Jump and Levine[18] provides a brief review of existing macroeconomic models in literature, their weaknesses, and future perspective. Cherrier, Duarte and Saïdi[19] trace the rise of heterogeneous household models in mainstream macroeconomics from the turn of the 1980s to the early 2000s. They show that different communities across the US and Europe considered heterogeneous agents for various reasons and developed models that differed in their theoretical and empirical strategies.

The main contribution of this paper is that it provides a foundation for modeling an interaction of number of markets in an economy. This paper captures an interaction of three markets, i.e., goods, labor, and capital (Figure 1); however, the methodology can be extended to as many markets as required depending on objectives regarding prediction of various markets parameters to be achieved. The model captures not only both initial and final sets of equilibria before and after shocks to all markets, rather it predicts the full adjustment path of all markets from initial to final equilibriums after various shocks to one or more markets. The adjustment mechanism of the markets from one equilibrium to the other (i.e., a state of disequilibrium) after a shock is based on lack of coordination among economic agents at existing prices and lack of information regarding the magnitude and direction of shocks happening to various markets.

The remainder of this paper is organized as follows: Section 2 explains the model based on three interacting markets. Section 3 provides a solution of the model in time domain when markets have no production frictions and price rigidities. Section 4 summarizes the findings and concludes. Section 5 explains future research prospects.

2 THE MODEL

Suppose there are three perfectly competitive markets, i.e., a single homogenous goods market, a money market, and a labor market interacting with each other. It will be demonstrated that the model is robust to heterogeneity and bounded rationality of economic agents. When a shock happens to one of the markets or few or all of them, there is feedback to all markets as they are interacting with each other and do not exist in isolation or independent of each other. The adjustment mechanism of the markets from one equilibrium to the other (i.e., a state of disequilibrium) after a shock is based on lack of coordination among economic agents at existing prices and lack of information regarding the magnitude and direction of shocks happening to various markets. Mathematical expressions representing economic agents' individual decisions are derived and solved simultaneously to capture collective outcome of interaction of all markets.

2.1 Goods Market

We assume a perfectly competitive goods (a single homogeneous commodity) market in equilibrium. Infinitely lived economic agents include a representative -or a unit mass of- producer who has a production technology with inputs of labor and capital which he/she employs to produce goods/output; a middleman who makes purchases from producer, stores goods/output (holds, maintains and manages an inventory) and sells goods to consumer at market price; and a representative –or a unit mass of– consumer who buys goods from middleman, accumulates capital by investing and supplies labor inelastically.

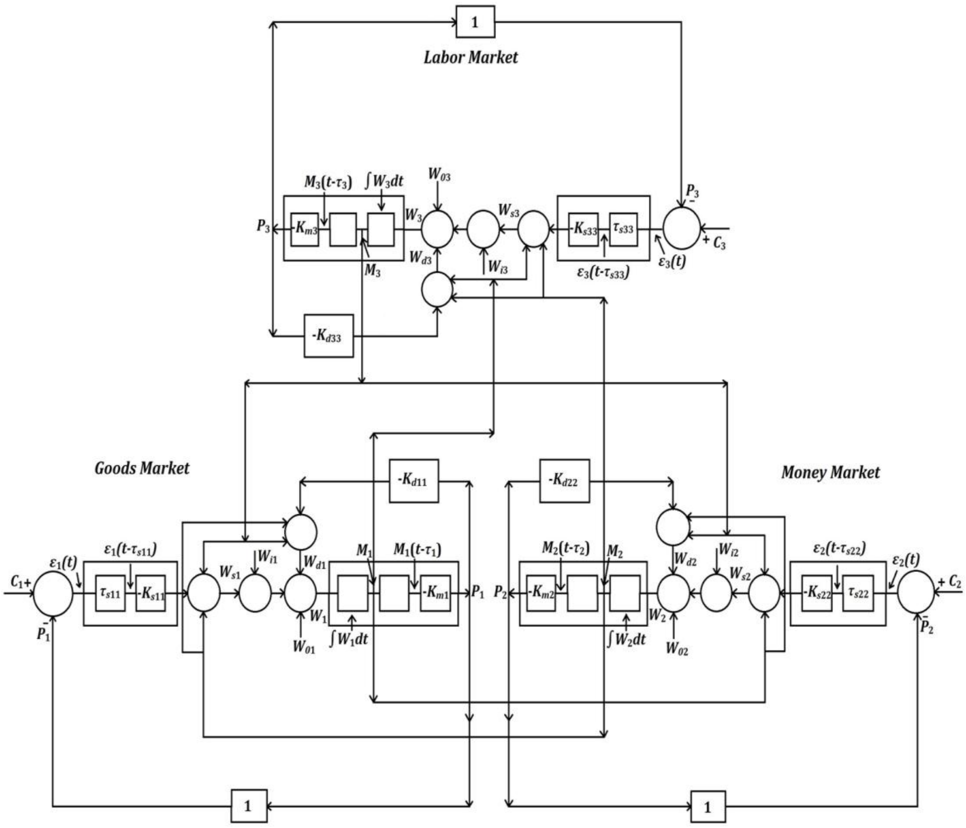

Figure 1. A Theoretical Dynamical Macroeconomic Model based on Interacting Markets.

Producer is a price taker. Middleman maximizes profit and pays a price with as the market price for goods, and to producer. Middleman is also a price taker when market is in equilibrium, however, he/she has an incentive to change price when market goes out of equilibrium after an exogenous shock, until market attains a new steady state equilibrium where the middleman again becomes a price taker as he/she would lose business by deviating from a price equals marginal cost. Price adjustment mechanism is based on the fact that after a shock puts market out of equilibrium, economic agents' decisions lack coordination at existing price in addition to a lack of information about exact magnitude of the shock, and direction and magnitude of future shocks (if any). A shock to one of the three markets (or shocks to all markets) might be exogenous, however, the feedback mechanism leads parameters in all markets adjust endogenously until final steady state equilibrium arrives. Following example illustrates how goods market performs as a dynamical system. Consider a goods market in a steady state equilibrium, where producer and consumer have a steady state rate of production and consumption respectively, and middleman holds a steady state equilibrium stock of inventory. Suppose a negative demand shock happens to the market, which will lead the size of inventory with middleman to grow. This will modify middleman's profit maximizing condition due to which middleman will change (decrease) price in direction of bringing new/final equilibrium. The lower price than before will modify producer and consumer's profit and utility maximizing conditions respectively, due to which they will modify their economic decisions, which will further impact inventory, and middleman will decrease price further to bring market closer to final equilibrium. This endogenous decision making by all agents continues until final steady state equilibrium is attained with a lower price and output as compared to those in initial equilibrium. Steady state equilibrium is defined as given below:

(i) Economic agents do not have an incentive to change their behavior unless an exogenous shock happens and modifies their incentive.

(ii) Rate of production/supply equals rate of consumption/demand, and size of inventory stays the same.

Routh-Hurwitz stability criterion provides a necessary and sufficient condition for stability of a linear dynamical system, which implies after an exogenous shock/input to dynamical system, it drifts toward final steady state and ultimately arrives there unless another shock hits the market. However, if more shocks keep happening, and market is stable based on Routh-Hurwitz stability criterion, it will always drift toward a steady state equilibrium which will be decided by values of parameters in final steady state value expression.

A steady state equilibrium is sustainable only if economic agents do not have an incentive to deviate from their responses, which implies agents stick to their behavior unless an exogenous shock hits the market. Similarly, if an exogenous shock happens to the market, it disturbs at least one of the variables from steady state equilibrium value, which influences agents' decision making by modifying the information their decisions are based on. For example, if a negative demand shock happens to the market, middleman's inventory starts getting piled up as the consumption/demand rate has decreased, whereas production/supply rate is the same as before. Now, middleman has an incentive to deviate from his equilibrium behavior as accumulation of inventory to an infinite extent is not sustainable for him. He will make a modified decision in his self-interest, e.g., by decreasing price and/or buying fewer goods from producer to avoid accumulating inventory beyond his financial/logistic capacity. Middleman's actions will interact with decisions of other agents, and hence they will modify their behaviors too. If market system is stable, the feedback loop of agents' decisions will come into play endogenously continuously affecting agents' actions until market attains a new steady state equilibrium (which it will be due to its stability like other stable dynamical systems), where again agents will not find it beneficial to deviate from their behavior. This shows that agents only modify their actions during market adjustment and continue with their responses during steady state equilibrium. To handle and solve complexity of dynamical market system, we present agents' behaviors in mathematical terms and to capture collective outcome of their individual actions, we solve mathematical expressions depicting their individual behaviors simultaneously. We use linearization of supply and demand curves where it is reasonable to do so, e.g., if initial and final equilibriums are not too far from each other, linearization seems to be a good approximation, however, highly non-linear demand and supply curves with initial and final equilibriums far from each other do not warrant linearization around steady state equilibrium, which will require modeling a non-linear dynamical system (beyond the scope of this research).

2.1.1 Middleman of Goods Market

Middleman maximizes profit by purchasing goods from producer and selling those to consumer. Although, it is in the best interest of middleman to buy and sell the same quantity to minimize cost, however, he/she does not have information regarding exact demand of goods, therefore, he holds an inventory by purchasing from producer for subsequently selling to consumer so that goods are readily available to be sold whenever there is a demand. Inventory of goods in a market reflects the difference of production and consumption; a change in inventory implies demand and/or supply rates are changing at different rates, whereas in a steady state equilibrium, supply and demand rates are the same and inventory size stays the same. Following expression for middleman's dynamic optimization problem has been derived in Nawaz[21]:

$$P_1 = -K_{m1} \int W_1 dt = -K_{m1} M_1(t) \quad (1)$$

Where

P_1 = price change,

K_{m1} = proportionality constant,

W_1 = supply rate − demand rate,

$M_1 = m_1 - m_{1s}$ = change in inventory in the market.

As there is a time delay involved between an inventory change and a price change due to price rigidity, incorporating a dead time element, the above expression becomes:

$$P_1 = -K_{m1} M_1(t - \tau_1) \quad (2)$$

There can be an input other than inventory affecting price which can get added to the above equation as follows:

$$P_1 = -K_{m1} M_1(t - \tau_1) + J_1 \quad (3)$$

2.1.2 Producer of Goods

As a rational agent, producer maximizes present discounted value of stream of profits for all future times having present value at $t = 0$ as under:

$$V(0) = \int_0^\infty \left[\alpha p_1(t) F\big(K(t), L(t)\big) - p_2(t) I(t) - p_3(t) L(t)\right] e^{-rt} dt \quad (4)$$

α being fraction of market price charged by producer to middleman; r being discount rate; $L(t)$ (labor) and $I(t)$ (level of investment) as *control variables* and $K(t)$ being *state variable*. p_2 is price of capital, and p_3 is wage/price of labor. Maximization problem is as under:

$$\underset{\{L(t),I(t)\}}{Max} V(0) = \int_0^\infty \left[\alpha p_1(t)F\big(K(t),L(t)\big) - p_2(t)I(t) - p_3(t)L(t)\right]e^{-rt}dt$$

subject to following constraints:
$\dot{K}(t) = I(t) - \delta K(t)$ (state equation, describing how state variable changes with time),
$K(0) = K_0$ (initial condition),
$K(t) \geq 0$ (non-negativity constraint on state variable),
$K(\infty)$ free (terminal condition).
Current-value Hamiltonian is as under:

$$\widetilde{H} = \alpha p_1(t)F\big(K(t),L(t)\big) - p_2(t)I(t) - p_3(t)L(t) + \mu(t)[I(t) - \delta K(t)] \ (5)$$

Maximizing conditions are as under:
(i) $L^*(t)$ and $I^*(t)$ maximize $\widetilde{H}$ for all t: $\frac{\partial \widetilde{H}}{\partial L} = 0$ and $\frac{\partial \widetilde{H}}{\partial I} = 0$,

(ii) $\dot{\mu} - r\mu = -\frac{\partial \widetilde{H}}{\partial K}$,

(iii) $\dot{K}^* = \frac{\partial H}{\partial \mu}$ (this just gives back the state equation),

(iv) $\lim_{t\to\infty} \mu(t)K(t)e^{-rt} = 0$ (the transversality condition).

Conditions (i) and (ii) can be expressed as follows:

$$\frac{\partial \widetilde{H}}{\partial L} = \alpha p_1(t)F_2^{'}\big(K(t),L(t)\big) - p_3(t) = 0 \ (6)$$

$$\frac{\partial \widetilde{H}}{\partial I} = - p_2(t) + \mu(t) = 0 \ (7)$$

And

$$\dot{\mu} - r\mu = -\frac{\partial \widetilde{H}}{\partial K} = -\left[\alpha p_1(t)F_1^{'}\big(K(t),L(t)\big) - \delta\mu(t)\right] \ (8)$$

Substituting values of $\dot{\mu}$ and μ from Equation (7) in (8) yields

$$\alpha p_1(t)F_1^{'}\big(K(t),L(t)\big) - (r+\delta)p_2(t) + \dot{p_2}(t) = 0 \ (9)$$

If $p_1(t)$ (price of goods) increases, producer faces following inequalities at existing level of investment and labor:

$$\alpha p_1(t)F_2^{'}\big(K(t),L(t)\big) - p_3(t) > 0,$$

$$\alpha p_1(t)F_1^{'}\big(K(t),L(t)\big) - (r+\delta)p_2(t) + \dot{p_2}(t) > 0$$

Similarly, if either inventory of capital or labor goes up (ceteris paribus), producer faces following inequalities:

$$\alpha p_1(t)F_2^{'}\big(K(t),L(t)\big) - p_3(t) > 0,$$

$$\alpha p_1(t)F_1^{'}\big(K(t),L(t)\big) - (r+\delta)p_2(t) + \dot{p_2}(t) > 0$$

Therefore, if price of goods, inventory of capital or labor goes up (ceteris paribus), producer will increase production, which implies (partially this has already been discussed in Nawaz[21])

$$W_{s1}(Goods) = - K_{s11}\varepsilon_1(t-\tau_{s11}) + K_{s12}f_{s12}\varkappa M_2(t-\tau_{s12}) + K_{s13}M_3(t-\tau_{s13}) \ (10)$$

Where
W_{s1} = Change in production of goods by producer,

$\varepsilon_1(t) = (c_1 - c_{1s}) - (p_1 - p_{1s}) = C_1 - P_1$,

$M_2(t)$ = Inventory of money/capital in economy,

$M_3(t) = \text{Inventory/stock of unsold labor/leisure in economy},$

p_{1s} and c_{1s} are steady state values; τ_{s11}, τ_{s12}, and τ_{s13} are dead time; c_1=a reference price (such as retail price which includes production cost, profit of producer and profit of middleman); K_{s11}, K_{s12} and K_{s13} are proportionality constants, f_{s12} is fraction of $\varkappa M_2$ (change in money/savings/investment; $\varkappa M_2$ is fraction of increased money supply which is saved and deposited into banks by households and hence is available to be given as loans, it does not include the cash spent by households, consumers, producers, etc., to buy goods, which is written as $(1-\varkappa)M_2$) which went into production of goods.

Robustness of Equation (10) with regard to heterogeneity and bounded rationality has been checked in Appendix. Fiscal policy and the feedback of the fiscal policy on demand and supply can be captured as follows: If an ad valorem consumption tax T is imposed on buyer, the market price the buyer will be paying will be inclusive of the consumption tax, however, price consideration for producer's decision making regarding how much to produce will be the one before tax, i.e.,

$$\varepsilon_1(t) = Tp_1 - P_1$$

If there was an exporter supplying production to the domestic market along with the local producer, the total change in supply would get bifurcated as follows:

$$W_{s1} =- K_{s11\text{d}}[C_{1\text{d}}(t) - P_1(t)] - K_{s11\text{e}}[C_{1\text{e}}(t) - P_1(t)]$$

where the subscripts d and e denote the domestic producer and the exporter (foreign producer) in the foreign country respectively. If the government imposes a per unit tariff on the imports at $t = 0$, the following expressions would be substituted for modeling the tariff policy:

$$C_{1\text{d}}(t) = 0, C_{1\text{e}}(t) = T.$$

2.1.3 Consumer of Goods

As a rational agent, consumer maximizes present discounted value of stream of utilities for all future times having present value at $t = 0$ as under:

$$V(0) = \int_0^\infty U(x(t))e^{-\rho t}dt \quad (11)$$

ρ being discount rate; and $x(t)$ (consumption) as *control variable*. Utility comparisons are being made across time. In a typical intertemporal consumption model, the summation of utilities discounted from various future times would be maximized with respect to the amounts consumed in each period, subject to an intertemporal budget constraint that says that the present value of current and future expenditures does not exceed the present value of financial resources available for spending. Despite arguments about how discount factor should be interpreted, the basic idea is that all other things equal, the agent prefers to have something now as opposed to later. Maximization problem is as under:

$$\underset{\{x(t)\}}{Max} V(0) = \int_0^\infty U(x(t))e^{-\rho t}dt,$$

subject to following constraints:

$\dot{a}(t) = \beta p_2(t)a(t) + p_3(t)L(t) - p_1(t)x(t)$ (state equation, describing how state variable changes with time). β is fraction of $p_2(t)$ charged by households to financial intermediaries, $a(t)$ is asset holdings (a *state variable*) and $p_3(t)$ and $p_2(t)$ are time path of wages and return on assets. $a(0) = a_s$ (initial condition), $a(t) \geq 0$ (non-negativity constraint on state variable), $a(\infty)$ free (terminal condition). Current-value Hamiltonian is as under:

$$\widetilde{H} = U(x(t)) + \mu(t)[\beta p_2(t)a(t) + p_3(t)L(t) - p_1(t)x(t)] \quad (12)$$

Maximizing conditions are as under:

(i) $x^*(t)$ maximizes $\widetilde{H}$ for all t: $\frac{\partial \widetilde{H}}{\partial x} = 0$,

(ii) $\dot{\mu} - \rho\mu =- \frac{\partial \widetilde{H}}{\partial a}$,

(iii) $\dot{a}^* = \frac{\partial H}{\partial \mu}$ (this just gives back the state equation),

(iv) $\lim\limits_{t\to\infty} \mu(t)a(t)e^{-\rho t} = 0$ (the transversality condition).

Conditions (i) and (ii) can be expressed as follows:

$$\frac{\partial \widetilde{H}}{\partial x} = U^{'}(x(t)) - \mu(t)p_1(t) = 0 \ (13)$$

And

$$\dot{\mu} - \rho\mu = -\frac{\partial \widetilde{H}}{\partial a} = -\mu(t)\beta p_2(t) \ (14)$$

If price of good x goes up, consumer faces (at previous level of consumption) following inequality:

$$\frac{\partial \widetilde{H}}{\partial x} = U^{'}(x(t)) - \mu(t)p_1(t) < 0$$

To satisfy condition of dynamic optimization after price increase, consumer will decrease consumption of good x. If inventory of unsold labor goes up, production of x by producer goes up which brings price of x down after a time delay, and hence consumption of x increases. If inventory of money/capital goes up, price of capital goes down and consumer faces following inequality:

$$-\frac{\partial \widetilde{H}}{\partial a} = -\mu(t)\beta p_2(t) > \dot{\mu} - \rho\mu,$$

which implies consumer will reduce purchase of assets and will increase consumption of $x(t)$. Above discussion implies,

$$W_{d1}(Goods) = -K_{d11}P_1 + K_{d121}f_{d12}\varkappa M_2(t) + K_{d122}(1-\varkappa)M_2(t) + K_{d13}M_3(t-\tau_{d13}) \ (15)$$

Where

$$W_{d1} = \text{Change in consumption of goods by consumer},$$

τ_{d13} is dead time; K_{d11}, K_{d121}, K_{d122} and K_{d13} are proportionality constants, f_{d12} is fraction of $\varkappa M_2$ (change in money/savings/investment; $\varkappa M_2$ is fraction of increased money supply which is saved and deposited into banks by households and hence is available to be given as loans, it does not include the cash spent by households, consumers, producers, etc., to buy goods, which is written as $(1-\varkappa)M_2$) which went into consumption of goods, e.g., people get loan to buy consumption goods, e.g., cars, etc. Robustness of Equation (15) with regard to heterogeneity and bounded rationality has been checked in Appendix.

2.2 Money/Capital Market

The model for money/capital market has already been discussed in Nawaz[22]. Infinitely lived economic agents include a representative -or a unit mass of- producer of funds, i.e., household, who saves money and have that deposited in commercial banks for a return on assets; a financial intermediary/commercial bank, who buys/borrows funds from household and sells/lends to producer/firm of goods; and a representative –or a unit mass of– consumer of funds, i.e., firm/producer of goods who borrows from commercial bank. Commercial bank holds deposits/inventory of funds to be sold to producer/firm.

2.2.1 Financial Intermediary/Commercial Bank

Commercial bank maximizes profit by purchasing/borrowing funds from producer of funds, i.e., household and selling those to consumer, i.e., producer/firm. Although, it is in the best interest of middleman to buy and sell the same quantity to minimize cost, however, he/she does not have information regarding exact demand of funds, therefore, he holds an inventory by purchasing from producer for subsequently selling to consumer so that funds are readily available to be sold whenever there is a demand. Inventory of funds in money/capital market reflects difference of production and consumption; a change in inventory implies demand and/or supply rates are changing at different rates, whereas in a steady state equilibrium, supply and demand rates are the same and inventory size stays the same. Following expression for financial intermediary's dynamic optimization problem has been derived in Nawaz[22]:

$$P_2 = -K_{m2}\int W_2 dt = -K_{m2}M_2(t) \ (16)$$

Where

$$P_2 = \text{interest rate change},$$
$$K_{m2} = \text{proportionality constant},$$
$$W_2 = \text{supply rate} - \text{demand rate},$$
$$M_2 = m_2 - m_{2s} = \text{change in inventory of funds in financial/money/capital market}.$$

As there is a time delay involved between an inventory change and an interest rate change due to price rigidity, incorporating a dead time element, the above expression becomes:

$$P_2 = -K_{m2}M_2(t-\tau_2) \quad (17)$$

There can be an input other than inventory of funds affecting interest rate which can get added to the above equation as follows:

$$P_2 = -K_{m2}M_2(t-\tau_2) + J_2 \quad (18)$$

2.2.2 Household/Producer of Funds

Consumer of goods is producer of funds which has already been discussed in section 2.1.3. From Equation (14), if price of capital/interest rate goes up, household faces following inequality:

$$-\frac{\partial \widetilde{H}}{\partial a} = -\mu(t)\beta p_2(t) < \dot{\mu} - \rho\mu,$$

which implies after an interest rate increase, household increases production/supply of funds to financial intermediary. If inventory of goods and unsold labor goes up (which increases production of good x after a time delay), price of good x goes down, and household faces following expression:

$$\frac{\partial \widetilde{H}}{\partial x} = U^{'}(x(t)) - \mu(t)p_1(t) > 0,$$

and household will increase consumption of good x with fewer resources left for bank deposits which will go down. Above discussion implies following expression (the problem has partially been discussed in Nawaz[22]) for household production of funds (to be deposited in commercial bank as savings):

$$W_{s2}(Money/Capital) = -K_{s21}M_1 - K_{s22}\varepsilon_2(t-\tau_{s22}) + K_{s23}M_3(t-\tau_{s23}) \quad (19)$$

Where

W_{s2} = Change in production of funds by household,

$\varepsilon_2(t) = (c_2 - c_{2s}) - (p_2 - p_{2s}) = C_2 - P_2,$

p_{2s} and c_{2s} are steady state values; τ_{s22}, and τ_{s23} are dead time; c_2 = a reference price of assets (such as the yield on an asset which includes the cost of savings, profit of household and profit of financial intermediary); K_{s21}, K_{s22} and K_{s23} are proportionality constants. Robustness of Equation (19) with regard to heterogeneity and bounded rationality has been checked in Appendix.

2.2.3 Firm/Consumer of Funds

Producer of goods/firm is consumer of funds which has already been discussed in section 2.1.2. From Equation (9), if price of capital/interest rate goes up (ceteris paribus), firm faces following inequality:

$$\alpha p_1(t)F_1^{'}\big(K(t), L(t)\big) - (r+\delta)p_2(t) + \dot{p_2}(t) < 0,$$

which implies after an interest rate increase, consumer of funds demands lower amount of funds/capital. If inventory of goods goes up, $p_1(t)$ (price of goods) decreases, and from Equations (6) and (9), producer faces following inequalities at existing level of investment and labor:

$$\alpha p_1(t)F_2^{'}\big(K(t), L(t)\big) - p_3(t) < 0,$$

$$\alpha p_1(t)F_1^{'}\big(K(t), L(t)\big) - (r+\delta)p_2(t) + \dot{p_2}(t) < 0.$$

This implies that after an increase in goods inventory, producer demands a lower quantity of capital. Similarly, if inventory of unsold labor goes up, labor becomes cheaper, and demand of capital (which compliments labor) by firm goes up due to following inequality faced by firm:

$$\alpha p_1(t)F_2^{'}\big(K(t), L(t)\big) - p_3(t) > 0.$$

Above discussion implies the following expression:

$$W_{d2}(Money/Capital) = -K_{d21}M_1(t-\tau_{d21}) - K_{d22}P_2 + K_{d23}M_3(t-\tau_{d23}) \quad (20)$$

Where

$$W_{d2} = \text{Change in demand of money/capital by firm.}$$

τ_{d21}, and τ_{d23} are dead time; K_{d21}, K_{d22} and K_{d23} are proportionality constants. Robustness of Equation (20) with regard to heterogeneity and bounded rationality has been checked in Appendix.

2.3 Labor Market

Infinitely lived economic agents include a representative -or a unit mass of- producer (that produces labor-a private training institution in perfect competition with identical institutions, and demands input labor and capital), a worker/laborer (who buys labor skills from producers to sell to consumers, and enjoys some leisure), and a representative –or a unit mass of– consumer (who buys labor). Producer provides training/skills to worker/laborer, who enjoys some leisure and sells labor to consumer at market wage. In the model, the worker/laborer has a key role, as he sets wage $p_3(t)$ by maximizing the difference between the revenue for selling labor to consumers and the costs of leisure. The worker pays a price γp_3 ($\gamma < 1$) to producer for acquiring labor skills subsequently sold to consumer of labor and producer of labor skills is a price-taker.

2.3.1 Worker/Laborer

Worker/laborer buys labor skills (training) from producer and supplies labor to consumer of labor for getting income. Worker does not sell all his time as labor and rather also enjoys some leisure, which depends on wage rate. The proportion of his time as leisure can be adjusted by worker/laborer according to supply/demand of labor in the market. In case leisure stays the same in the market, it reflects that supply and demand rates for labor are equal. If leisure goes up or reduces, that means either there is a change in demand or supply of labor or both with unequal rates. Following expression for worker's dynamic optimization problem has been derived:

$$P_3 =- K_{m3} \int W_3 dt =- K_{m3} M_3(t) \quad (21)$$

Where
P_3 = wage change,
K_{m3} = proportionality constant,
W_3 = supply rate of labor − demand rate of labor,
m_3 = leisure at time t,
m_{3s} = leisure in steady state equilibrium,
$M_3 = m_3 - m_{3s}$ = change in leisure in labor market.

As there is a time delay involved between leisure change and wage change in labor market due to time involved in penetration of information, incorporating a dead time element, the above expression becomes:

$$P_3 =- K_{m3} M_3(t - \tau_3) \quad (22)$$

There can be an input other than leisure change affecting wage which can get added to the above equation as follows:

$$P_3 =- K_{m3} M_3(t - \tau_3) + J_3 \quad (23)$$

2.3.2 Producer of Labor

As a rational agent, producer of labor (a private training institution in perfect competition with identical institutions) maximizes present discounted value of stream of profits for all future times having present value at $t = 0$ as under:

$$V(0) = \int_0^{\infty} \left[\gamma p_3(t) F_l\big(K_l(t), L_l(t)\big) - p_2(t) I_l(t) - p_l(t) L_l(t)\right] e^{-r_l t} dt \quad (24)$$

γ being fraction of market price, i.e., p_3 charged by producer of labor to worker/laborer; r_l being discount rate; $L_l(t)$ (labor) and $I_l(t)$ (level of investment in terms of capital/funds/money with same price of capital as in goods market) as *control variables* and $K_l(t)$ being *state variable*. p_2 is price of capital, and p_l is wage/price of labor (this is input labor to produce type of skills/labor to be used in goods market). Maximization problem is as under:

$$\underset{\{L_l(t), I(t)\}}{Max} V(0) = \int_0^{\infty} \left[\gamma p_3(t) F_l\big(K_l(t), L_l(t)\big) - p_2(t) I_l(t) - p_l(t) L_l(t)\right] e^{-r_l t} dt,$$

subject to following constraints
$\dot{K}_l(t) = I_l(t) - \delta_l K_l(t)$ (state equation, describing how the state variable changes with time),

$K_l(0) = K_{l0}$ (initial condition),

$K_l(t) \geq 0$ (non-negativity constraint on state variable),

$K_l(\infty)$ free (terminal condition).

Current-value Hamiltonian is as under:

$$\widetilde{H} = \gamma p_3(t)F_l\big(K_l(t), L_l(t)\big) - p_2(t)I_l(t) - p_l(t)L_l(t) + \mu(t)[I_l(t) - \delta_l K_l(t)] \quad (25)$$

Maximizing conditions are as under:

(i) $L_l^*(t)$ and $I_l^*(t)$ maximize $\widetilde{H}$ for all t: $\frac{\partial \widetilde{H}}{\partial L_l} = 0$ and $\frac{\partial \widetilde{H}}{\partial I_l} = 0$,

(ii) $\dot{\mu} - r_l\mu = -\frac{\partial \widetilde{H}}{\partial K_l}$,

(iii) $\dot{K}_l^* = \frac{\partial H}{\partial \mu}$ (this just gives back the state equation),

(iv) $\lim_{t\to\infty} \mu(t)K_l(t)e^{-r_l t} = 0$ (the transversality condition).

Conditions (i) and (ii) can be expressed as follows:

$$\frac{\partial \widetilde{H}}{\partial L_l} = \gamma p_3(t)F_{l2}^{'}\big(K_l(t), L_l(t)\big) - p_l(t) = 0 \quad (26)$$

$$\frac{\partial \widetilde{H}}{\partial I_l} = - p_2(t) + \mu(t) = 0 \quad (27)$$

And

$$\dot{\mu} - r_l\mu = -\frac{\partial \widetilde{H}}{\partial K_l} = -\left[\gamma p_3(t)F_{l1}^{'}\big(K_l(t), L_l(t)\big) - \delta_l\mu(t)\right] \quad (28)$$

Substituting values of $\dot{\mu}$ and μ from Equation (27) in (28) yields

$$\gamma p_3(t)F_{l1}^{'}\big(K_l(t), L_l(t)\big) - (r_l + \delta_l)p_2(t) + \dot{p_2}(t) = 0 \quad (29)$$

If $p_3(t)$ (price of labor) increases, producer of labor faces following inequalities at existing level of investment and labor:

$$\gamma p_3(t)F_{l2}^{'}\big(K_l(t), L_l(t)\big) - p_l(t) > 0,$$

$$\gamma p_3(t)F_{l1}^{'}\big(K_l(t), L_l(t)\big) - (r_l + \delta_l)p_2(t) + \dot{p_2}(t) > 0$$

This implies that producer of labor increases production as the market price of their output (labor skills) increases. If inventory of goods (x) increases, price of goods goes down, which decreases production of goods and hence demand of labor, which will reduce price of labor used in goods production leading to a lower supply of labor due to following inequality faced by producer of labor:

$$\gamma p_3(t)F_{l2}^{'}\big(K_l(t), L_l(t)\big) - p_l(t) < 0$$

Similarly, if inventory of capital goes up (ceteris paribus), producer of labor faces following inequality and increases production of labor:

$$\gamma p_3(t)F_{l1}^{'}\big(K_l(t), L_l(t)\big) - (r_l + \delta_l)p_2(t) + \dot{p_2}(t) > 0$$

Above discussion implies the following expression:

$$W_{s3}(Labor) = - K_{s31}M_1(t - \tau_{s31}) + K_{s32}f_{s32}\varkappa M_2(t - \tau_{s32}) - K_{s33}\varepsilon_3(t - \tau_{s33}) \quad (30)$$

Where

W_{s3} = Change in production of labor by producer,

$\varepsilon_3(t) = (c_3 - c_{3s}) - (p_3 - p_{3s}) = C_3 - P_3$,

$M_1(t)$ = Inventory of goods in economy,

$M_2(t)$ = Inventory of money/capital in economy,

p_{3s} and c_{3s} are steady state values; τ_{s31}, τ_{s32}, and τ_{s33} are dead time; c_3 = a reference price (such as retail price which includes production cost of labor, profit of producer and profit of worker/laborer); K_{s31}, K_{s32} and K_{s33} are proportionality constants. f_{s32} is fraction of $\varkappa M_2$ which went into increased supply/production of labor. Robustness of Equation (30) with regard to heterogeneity and bounded rationality has been checked in Appendix.

2.3.3 Firm/Consumer of Labor

Producer of goods/firm is consumer of labor which has already been discussed in section 2.1.2. From Equation (6), if price of labor/wage goes up (ceteris paribus), firm faces following inequality:

$$\alpha p_1(t)F_2^{'}\big(K(t), L(t)\big) - p_3(t) < 0,$$

which implies after a wage increase, consumer of labor demands lower quantity of labor. If inventory of goods goes up, $p_1(t)$ (price of goods) decreases, and from Equation (6) and (9), firm/producer of goods (x) faces following inequalities at existing level of investment and labor:

$$\alpha p_1(t)F_2^{'}\big(K(t), L(t)\big) - p_3(t) < 0,$$

$$\alpha p_1(t)F_1^{'}\big(K(t), L(t)\big) - (r+\delta)p_2(t) + \dot{p_2}(t) < 0.$$

This implies that after an increase in goods inventory, producer of goods demands a lower quantity of labor. Similarly, if inventory of money goes up, capital becomes cheaper, and demand of labor (which compliments capital) by firm goes up due to following inequality faced by firm

$$\alpha p_1(t)F_1^{'}\big(K(t), L(t)\big) - (r+\delta)p_2(t) + \dot{p_2}(t) > 0.$$

Above discussion implies the following expression:

$$W_{d3}(Labor) =- K_{d31}M_1(t-\tau_{d31}) + K_{d32}f_{d32}\varkappa M_2(t-\tau_{d32}) - K_{d33}P_3 \quad (31)$$

Where

W_{d3} = Change in demand of labor by firm.

$$f_{s12} + f_{s32} + f_{d12} + f_{d32} = 1$$

τ_{d31}, and τ_{d32} are dead time; K_{d31}, K_{d32} and K_{d33} are proportionality constants. Robustness of Equation (31) with regard to heterogeneity and bounded rationality has been checked in Appendix.

3 SOLUTION OF THE MODEL WITH FLEXIBLE PRICES AND NO PRODUCTION FRICTION

Solution of the model is presented for the simplest case, i.e., when all dead times are set to zero. From Equations (1), (21) and (26), we have following expressions:

$$\frac{dP_1(t)}{dt} =- K_{m1}W_1(t),$$

$$\frac{dP_2(t)}{dt} =- K_{m2}W_2(t),$$

$$\frac{dP_3(t)}{dt} =- K_{m3}W_3(t),$$

Where

$$W_1(t) = W_{i1}(t) - W_{01}(t) + W_{s1}(t) - W_{d1}(t) \equiv D_1(t) + W_{s1}(t) - W_{d1}(t),$$

$$W_2(t) = W_{i2}(t) - W_{02}(t) + W_{s2}(t) - W_{d2}(t) \equiv D_2(t) + W_{s2}(t) - W_{d2}(t),$$

$$W_3(t) = W_{i3}(t) - W_{03}(t) + W_{s3}(t) - W_{d3}(t) \equiv D_3(t) + W_{s3}(t) - W_{d3}(t)$$

$D_1(t)$, $D_2(t)$, and $D_3(t)$ club together exogenous supply and demand shocks in goods, capital, and labor markets respectively. Substituting the values of $W_1(t)$, $W_2(t)$, and $W_3(t)$ in above differential equations, we obtain:

$$\frac{dP_1(t)}{dt} =- K_{m1}[D_1(t) + W_{s1}(t) - W_{d1}(t)] \quad (32)$$

$$\frac{dP_2(t)}{dt} =- K_{m2}[D_2(t) + W_{s2}(t) - W_{d2}(t)] \quad (33)$$

$$\frac{dP_3(t)}{dt} =- K_{m3}[D_3(t) + W_{s3}(t) - W_{d3}(t)] \quad (34)$$

Where

$$W_{s1}(Goods) = -K_{s11}\varepsilon_1(t-\tau_{s11}) + K_{s12}f_{s12}\varkappa M_2(t-\tau_{s12}) + K_{s13}M_3(t-\tau_{s13}),$$
$$W_{s2}(Money/Capital) = -K_{s21}M_1 - K_{s22}\varepsilon_2(t-\tau_{s22}) + K_{s23}M_3(t-\tau_{s23}),$$
$$W_{s3}(Labor) = K_{s31}M_1(t-\tau_{s31}) + K_{s32}f_{s32}\varkappa M_2(t-\tau_{s32}) - K_{s33}\varepsilon_3(t-\tau_{s33})$$

$$W_{d1}(Goods) = -K_{d11}P_1 + K_{d121}f_{d12}\varkappa M_2(t) + K_{d122}(1-\varkappa)M_2(t) + K_{d13}M_3(t-\tau_{d13}),$$
$$W_{d2}(Money/Capital) = -K_{d21}M_1(t-\tau_{d21}) - K_{d22}P_2 + K_{d23}M_3(t-\tau_{d23}),$$
$$W_{d3}(Labor) = -K_{d31}M_1(t-\tau_{d31}) + K_{d32}f_{d32}\varkappa M_2(t-\tau_{d32}) - K_{d33}P_3,$$

Substituting values of $W_{s1}(t)$, $W_{s2}(t)$, $W_{s3}(t)$, $W_{d1}(t)$, $W_{d2}(t)$, and $W_{d3}(t)$ in Equations (32)-(34) leads to following expressions:

$$\frac{dP_1(t)}{dt} = -K_{m1}\begin{bmatrix} D_1(t) - K_{s11}\varepsilon_1(t-\tau_{s11}) + K_{s12}f_{s12}\varkappa M_2(t-\tau_{s12}) + K_{s13}M_3(t-\tau_{s13}) \\ + K_{d11}P_1 - K_{d121}f_{d12}\varkappa M_2(t) - K_{d122}(1-\varkappa)M_2(t) - K_{d13}M_3(t-\tau_{d13}) \end{bmatrix} \quad (35)$$

$$\frac{dP_2(t)}{dt} = -K_{m2}\begin{bmatrix} D_2(t) - K_{s21}M_1 - K_{s22}\varepsilon_2(t-\tau_{s22}) + K_{s23}M_3(t-\tau_{s23}) \\ + K_{d21}M_1(t-\tau_{d21}) + K_{d22}P_2 - K_{d23}M_3(t-\tau_{d23}) \end{bmatrix} \quad (36)$$

$$\frac{dP_3(t)}{dt} = -K_{m3}\begin{bmatrix} D_3(t) + K_{s31}M_1(t-\tau_{s31}) + K_{s32}f_{s32}\varkappa M_2(t-\tau_{s32}) - K_{s33}\varepsilon_3(t-\tau_{s33}) \\ + K_{d31}M_1(t-\tau_{d31}) - K_{d32}f_{d32}\varkappa M_2(t-\tau_{d32}) + K_{d33}P_3 \end{bmatrix} \quad (37)$$

When all dead times are set to zero, above expressions become as follows:

$$\frac{dP_1(t)}{dt} = -K_{m1}\begin{bmatrix} D_1(t) - K_{s11}\varepsilon_1(t) + K_{s12}f_{s12}\varkappa M_2(t) + K_{s13}M_3(t) \\ + K_{d11}P_1 - K_{d121}f_{d12}\varkappa M_2(t) - K_{d122}(1-\varkappa)M_2(t) - K_{d13}M_3(t) \end{bmatrix} \quad (38)$$

$$\frac{dP_2(t)}{dt} = -K_{m2}\begin{bmatrix} D_2(t) - K_{s21}M_1 - K_{s22}\varepsilon_2(t) + K_{s23}M_3(t) \\ + K_{d21}M_1(t) + K_{d22}P_2 - K_{d23}M_3(t) \end{bmatrix} \quad (39)$$

$$\frac{dP_3(t)}{dt} = -K_{m3}\begin{bmatrix} D_3(t) + K_{s31}M_1(t) + K_{s32}f_{s32}\varkappa M_2(t) - K_{s33}\varepsilon_3(t) \\ + K_{d31}M_1(t) - K_{d32}f_{d32}\varkappa M_2(t) + K_{d33}P_3 \end{bmatrix} \quad (40)$$

Substituting values of $\varepsilon' s$, in above expressions, we get:

$$\frac{dP_1(t)}{dt} = -K_{m1}\begin{bmatrix} D_1(t) - K_{s11}\{C_1(t) - P_1(t)\} - K_{s12}f_{s12}\varkappa\frac{P_2(t)}{K_{m2}} - K_{s13}\frac{P_3(t)}{K_{m3}} \\ + K_{d11}P_1 + K_{d121}f_{d12}\varkappa\frac{P_2(t)}{K_{m2}} + K_{d122}(1-\varkappa)\frac{P_2(t)}{K_{m2}} + K_{d13}\frac{P_3(t)}{K_{m3}} \end{bmatrix} \quad (41)$$

$$\frac{dP_2(t)}{dt} = -K_{m2}\begin{bmatrix} D_2(t) + K_{s21}\frac{P_1(t)}{K_{m1}} - K_{s22}\{C_2(t) - P_2(t)\} - K_{s23}\frac{P_3(t)}{K_{m3}} \\ - K_{d21}\frac{P_1(t)}{K_{m1}} + K_{d22}P_2 + K_{d23}\frac{P_3(t)}{K_{m3}} \end{bmatrix} \quad (42)$$

$$\frac{dP_3(t)}{dt} = -K_{m3}\begin{bmatrix} D_3(t) - K_{s31}\frac{P_1(t)}{K_{m1}} - K_{s32}f_{s32}\varkappa\frac{P_2(t)}{K_{m2}} - K_{s33}\{C_3(t) - P_3(t)\} \\ - K_{d31}\frac{P_1(t)}{K_{m1}} + K_{d32}f_{d32}\varkappa\frac{P_2(t)}{K_{m2}} + K_{d33}P_3 \end{bmatrix} \quad (43)$$

Substituting values of $C' s$ equal to zero in Equations (41)-(43), we obtain:

$$\frac{dP_1(t)}{dt} = -K_{m1}\begin{bmatrix} D_1(t) + K_{s11}P_1(t) - K_{s12}f_{s12}\varkappa\frac{P_2(t)}{K_{m2}} - K_{s13}\frac{P_3(t)}{K_{m3}} \\ + K_{d11}P_1(t) + K_{d121}f_{d12}\varkappa\frac{P_2(t)}{K_{m2}} + K_{d122}(1-\varkappa)\frac{P_2(t)}{K_{m2}} + K_{d13}\frac{P_3(t)}{K_{m3}} \end{bmatrix},$$

$$= -\frac{K_{m1}}{K_{m2}K_{m3}}\begin{bmatrix} K_{m2}K_{m3}D_1(t) + K_{m2}K_{m3}\{K_{s11} + K_{d11}\}P_1(t) \\ + K_{m3}\{K_{d121}f_{d12}\varkappa - K_{s12}f_{s12}\varkappa + K_{d122}(1-\varkappa)\}P_2(t) + K_{m2}\{K_{d13} - K_{s13}\}P_3(t) \end{bmatrix},$$

$$\frac{dP_2(t)}{dt} = -K_{m2}\begin{bmatrix} D_2(t) + K_{s21}\frac{P_1(t)}{K_{m1}} + K_{s22}P_2(t) - K_{s23}\frac{P_3(t)}{K_{m3}} \\ - K_{d21}\frac{P_1(t)}{K_{m1}} + K_{d22}P_2(t) + K_{d23}\frac{P_3(t)}{K_{m3}} \end{bmatrix},$$

$$= -\frac{K_{m2}}{K_{m1}K_{m3}}\begin{bmatrix} K_{m1}K_{m3}D_2(t) + K_{m3}\{K_{s21} - K_{d21}\}P_1(t) + K_{m1}K_{m3}\{K_{s22} + K_{d22}\}P_2(t) \\ + K_{m1}\{K_{d23} - K_{s23}\}P_3(t) \end{bmatrix},$$

$$\frac{dP_3(t)}{dt} = -K_{m3}\begin{bmatrix} D_3(t) - K_{s31}\frac{P_1(t)}{K_{m1}} - K_{s32}f_{s32}\varkappa\frac{P_2(t)}{K_{m2}} + K_{s33}P_3(t) \\ - K_{d31}\frac{P_1(t)}{K_{m1}} + K_{d32}f_{d32}\varkappa\frac{P_2(t)}{K_{m2}} + K_{d33}P_3(t) \end{bmatrix},$$

$$-\frac{K_{m3}}{K_{m1}K_{m2}}\begin{bmatrix} K_{m1}K_{m2}D_3(t) - K_{m2}\{K_{s31} + K_{d31}\}P_1(t) + K_{m1}\varkappa\{K_{d32}f_{d32} - K_{s32}f_{s32}\}P_2(t) \\ + K_{m1}K_{m2}\{K_{s33} + K_{d33}\}P_3(t) \end{bmatrix}$$

The above nonhomogeneous linear system of differential equtions can be written as follows:

$$\frac{dP_1(t)}{dt} = -\frac{K_{m1}}{K_{m2}K_{m3}}\begin{bmatrix} K_{m2}K_{m3}D_1(t) + K_{m2}K_{m3}\{K_{s11} + K_{d11}\}P_1(t) \\ + K_{m3}\{K_{d121}f_{d12}\varkappa - K_{s12}f_{s12}\varkappa + K_{d122}(1-\varkappa)\}P_2(t) + K_{m2}\{K_{d13} - K_{s13}\}P_3(t) \end{bmatrix} \quad (44)$$

$$\frac{dP_2(t)}{dt} = -\frac{K_{m2}}{K_{m1}K_{m3}}\begin{bmatrix} K_{m1}K_{m3}D_2(t) + K_{m3}\{K_{s21} - K_{d21}\}P_1(t) + K_{m1}K_{m3}\{K_{s22} + K_{d22}\}P_2(t) \\ + K_{m1}\{K_{d23} - K_{s23}\}P_3(t) \end{bmatrix} \quad (45)$$

$$\frac{dP_3(t)}{dt} = -\frac{K_{m3}}{K_{m1}K_{m2}}\begin{bmatrix} K_{m1}K_{m2}D_3(t) - K_{m2}\{K_{s31} + K_{d31}\}P_1(t) + K_{m1}\varkappa\{K_{d32}f_{d32} - K_{s32}f_{s32}\}P_2(t) \\ + K_{m1}K_{m2}\{K_{s33} + K_{d33}\}P_3(t) \end{bmatrix} \quad (46)$$

If $D_1(t)$, $D_2(t)$, and $D_3(t)$ get step inputs, i.e., A_1, A_2, and A_3 respectively, Equations (44)-(46) can be written as follows:

$$\frac{dP_1(t)}{dt} = -\frac{K_{m1}}{K_{m2}K_{m3}}\begin{bmatrix} K_{m2}K_{m3}A_1 + K_{m2}K_{m3}\{K_{s11} + K_{d11}\}P_1(t) \\ + K_{m3}\{K_{d121}f_{d12}\varkappa - K_{s12}f_{s12}\varkappa + K_{d122}(1-\varkappa)\}P_2(t) + K_{m2}\{K_{d13} - K_{s13}\}P_3(t) \end{bmatrix} \quad (47)$$

$$\frac{dP_2(t)}{dt} = -\frac{K_{m2}}{K_{m1}K_{m3}}\begin{bmatrix} K_{m1}K_{m3}A_2 + K_{m3}\{K_{s21} - K_{d21}\}P_1(t) + K_{m1}K_{m3}\{K_{s22} + K_{d22}\}P_2(t) \\ + K_{m1}\{K_{d23} - K_{s23}\}P_3(t) \end{bmatrix} \quad (48)$$

$$\frac{dP_3(t)}{dt} = -\frac{K_{m3}}{K_{m1}K_{m2}}\begin{bmatrix} K_{m1}K_{m2}A_3 - K_{m2}\{K_{s31} + K_{d31}\}P_1(t) + K_{m1}\varkappa\{K_{d32}f_{d32} - K_{s32}f_{s32}\}P_2(t) \\ + K_{m1}K_{m2}\{K_{s33} + K_{d33}\}P_3(t) \end{bmatrix} \quad (49)$$

After estimating and substituting empirical parameter values in above system of differential equations and solving them, the response of three markets can be predicted after various kinds of shocks happen to one or more markets. Furthermore, optimal policies, such as optimal taxation, inflation control, trade, remittances, etc., affecting one or more of above markets subject to relevant constraints can be derived on the basis of above system of equations.

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The methodology employed above can be extended to n number of markets in an economy. In order to solve above system of differential equations, it can be written in matrix form as follows:

$$X^{'}(t) = \Theta(t)X(t) + \Delta(t),$$

Where

$$X^{'}(t) = \begin{bmatrix} \dfrac{dP_1(t)}{dt} \\ \dfrac{dP_2(t)}{dt} \\ \dfrac{dP_3(t)}{dt} \end{bmatrix},$$

$$X(t) = \begin{bmatrix} P_1(t) \\ P_2(t) \\ P_2(t) \end{bmatrix},$$

$$\Theta(t) = \begin{bmatrix} \theta_{11} & \theta_{12} & \theta_{13} \\ \theta_{21} & \theta_{22} & \theta_{23} \\ \theta_{31} & \theta_{32} & \theta_{33} \end{bmatrix}$$

$$= \begin{bmatrix} -K_{m1}\{K_{s11} + K_{d11}\} & -\dfrac{K_{m1}}{K_{m2}}\{K_{d121}f_{d12}\varkappa - K_{s12}f_{s12}\varkappa + K_{d122}(1-\varkappa)\} & -\dfrac{K_{m1}}{K_{m3}}\{K_{d13} - K_{s13}\} \\ -\dfrac{K_{m2}}{K_{m1}}\{K_{s21} - K_{d21}\} & -K_{m2}\{K_{s22} + K_{d22}\} & -\dfrac{K_{m2}}{K_{m3}}\{K_{d23} - K_{s23}\} \\ \dfrac{K_{m3}}{K_{m1}}\{K_{s31} + K_{d31}\} & -\dfrac{K_{m3}}{K_{m2}}\varkappa\{K_{d32}f_{d32} - K_{s32}f_{s32}\} & -K_{m3}\{K_{s33} + K_{d33}\} \end{bmatrix},$$

$$\Delta(t) = \begin{bmatrix} -K_{m1}A_1 \\ -K_{m2}A_2 \\ -K_{m3}A_3 \end{bmatrix}$$

THEOREM 1: If the vector-valued functions $\Theta(t)$ and $\Delta(t)$ are continuous over an open interval J contains $t = 0$; then the initial value problem

$$X^{'}(t) = \Theta(t)X(t) + \Delta(t),$$
$$X(t = 0) = X_0,$$

has a unique vector-values solution $X(t)$ that is defined on entire interval J for any given initial value X_0.

Let us summarize general steps to find a solution to initial value problem,

$$X^{'}(t) = \Theta(t)X(t) + \Delta(t),$$
$$X(t = 0) = X_0,$$

Step 1: Find the general solution $X_c = c_1X_1(t) + c_2X_2(t) + \ldots + c_nX_n(t)$, where $X_1(t), X_2(t), \ldots, X_n(t)$ are a set of linearly independent solutions to the associate homogeneous system,

$$X^{'}(t) = \Theta(t)X(t),$$

Step 2: Find a particular solution $X_p(t)$ to the nonhomogeneous system,

$$X^{'}(t) = \Theta(t)X(t) + \Delta(t).$$

Step 3: Set $X(t) = X_c(t) + X_p(t)$ and use equation $X(t = 0) = X_0$, to determine $c_1, c_2, \ldots, c_n$.

The characteristic polynomial of Θ given by

$$\phi(\lambda) = |\Theta - \lambda I|,$$

is a cubic polynomial of λ. Routh–Hurwitz stability criterion for a third order characteristic polynomial, $\phi(\lambda) = \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0$ requires all roots to lie in the open left half-plane, which is possible if and only if a_2, a_1, and a_0 are positive, and $a_2a_1 > a_0$. From Algebra, we know that $\phi(\lambda) = 0$, has either 3 distinct real solutions, or 2 distinct solutions and one is a double solution, or one real solution and 2 conjugate complex solutions, or a triple solution. The following theorem summarize the solution to the homogeneous system,

THEOREM 2: Let $\phi(\lambda)$ be the characteristic polynomial of Θ, for $X^{'}(t) = \Theta(t)X(t)$, Case 1: $\phi(\lambda) = 0$, has three distinct real solutions, λ_1, λ_2, and λ_3. Suppose

$$\mathrm{v}_1 = \begin{bmatrix} v_{11} \\ v_{21} \\ v_{31} \end{bmatrix}, \mathrm{v}_2 = \begin{bmatrix} v_{12} \\ v_{22} \\ v_{32} \end{bmatrix}, \text{and } \mathrm{v}_3 = \begin{bmatrix} v_{13} \\ v_{23} \\ v_{33} \end{bmatrix},$$

are associate eigenvectors (i.e., $\Theta\mathrm{v}_1 = \lambda_1\mathrm{v}_1$, $\Theta\mathrm{v}_2 = \lambda_2\mathrm{v}_2$, and $\Theta\mathrm{v}_3 = \lambda_3\mathrm{v}_3$), then the general solution is

$$X_c(t) = c_1\mathrm{v}_1 e^{\lambda_1 t} + c_2\mathrm{v}_2 e^{\lambda_2 t} + c_3\mathrm{v}_3 e^{\lambda_3 t},$$

and the fundamental matrix is

$$\Psi(t) = \begin{bmatrix} v_{11}e^{\lambda_1 t} & v_{12}e^{\lambda_2 t} & v_{13}e^{\lambda_3 t} \\ v_{21}e^{\lambda_1 t} & v_{22}e^{\lambda_2 t} & v_{23}e^{\lambda_3 t} \\ v_{31}e^{\lambda_1 t} & v_{32}e^{\lambda_2 t} & v_{33}e^{\lambda_3 t} \end{bmatrix}$$

Case 2: $\phi(\lambda) = 0$, has a double solutions λ_0. $\phi(\lambda) = (\lambda - \lambda_0)^2(\lambda - \lambda_1)$, and λ_0 has multiplicity 2. Let $\mathrm{v}_3 = \begin{bmatrix} v_{13} \\ v_{23} \\ v_{33} \end{bmatrix}$ is the eigenvector associated with λ_1. There can be two possibilities following this: (1) λ_0 has two linearly independent eigenvectors: $\mathrm{v}_1 = \begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix}$, and $\mathrm{v}_2 = \begin{bmatrix} v_{12} \\ v_{22} \end{bmatrix}$ are associate linearly independent eigenvectors. Then the general solution is

$$X_c(t) = (c_1\mathrm{v}_1 + c_2\mathrm{v}_2)e^{\lambda_0 t} + c_3\mathrm{v}_3 e^{\lambda_1 t},$$

and the fundamental matrix is

$$\Psi(t) = \begin{bmatrix} v_{11}e^{\lambda_0 t} & v_{12}e^{\lambda_0 t} & v_{13}e^{\lambda_1 t} \\ v_{21}e^{\lambda_0 t} & v_{22}e^{\lambda_0 t} & v_{23}e^{\lambda_1 t} \\ v_{31}e^{\lambda_0 t} & v_{32}e^{\lambda_0 t} & v_{33}e^{\lambda_1 t} \end{bmatrix}$$

(2) λ_0 has one eigenvector: $\mathrm{v}_1 = \begin{bmatrix} v_{11} \\ v_{21} \\ v_{31} \end{bmatrix}$ is the associated eigenvector with respect to λ_0 and $\mathrm{v}_2 = \begin{bmatrix} v_{12} \\ v_{22} \\ v_{32} \end{bmatrix}$ is a solution of $(\lambda_0 I - \Theta)\mathrm{v}_2 = \mathrm{v}_1$. Then the general solution is

$$X_c(t) = \{c_1\mathrm{v}_1 + c_2(t\mathrm{v}_1 + \mathrm{v}_2)\}e^{\lambda_0 t} + c_3\mathrm{v}_3 e^{\lambda_1},$$

and the fundamental solution matrix is

$$\Psi(t) = \begin{bmatrix} v_{11}e^{\lambda_0 t} & (v_{11}t + v_{12})e^{\lambda_0 t} & v_{13}e^{\lambda_1} \\ v_{21}e^{\lambda_0 t} & (v_{21}t + v_{22})e^{\lambda_0 t} & v_{23}e^{\lambda_1} \\ v_{31}e^{\lambda_0 t} & (v_{31}t + v_{32})e^{\lambda_0 t} & v_{33}e^{\lambda_1} \end{bmatrix}$$

Case 3: $\phi(\lambda) = 0$ has two conjugate complex solutions $a \pm bi$, and a real solution λ_1. Suppose $\mathrm{v} = \begin{bmatrix} v_{11} + v_{12}i \\ v_{21} + v_{22}i \\ v_{31} + v_{32}i \end{bmatrix}$ is the associated complex eigenvector with respect to $a + bi$, and $\mathrm{v}_3 = \begin{bmatrix} v_{13} \\ v_{23} \\ v_{33} \end{bmatrix}$ is the eigenvector associated with λ_1, then the general solution is

$$X_c(t) = [c_1\{\mathrm{v}_1\cos(bt) - \mathrm{v}_2\sin(bt)\} + c_2\{\mathrm{v}_2\cos(bt) + \mathrm{v}_1\sin(bt)\}]e^{at} + c_3\mathrm{v}_3 e^{\lambda_1},$$

and the fundamental solution matrix is

$$\Psi(t) = \begin{bmatrix} e^{at}\{v_{11}\cos(bt) - v_{12}\sin(bt)\} & e^{at}\{v_{12}\cos(bt) + v_{11}\sin(bt)\} & v_{13}e^{\lambda_1} \\ e^{at}\{v_{21}\cos(bt) - v_{22}\sin(bt)\} & e^{at}\{v_{22}\cos(bt) + v_{21}\sin(bt)\} & v_{23}e^{\lambda_1} \\ e^{at}\{v_{31}\cos(bt) - v_{32}\sin(bt)\} & e^{at}\{v_{32}\cos(bt) + v_{31}\sin(bt)\} & v_{33}e^{\lambda_1} \end{bmatrix}$$

Case 4: $\phi(\lambda) = 0$ has solution λ_0 with multiplicity 3, and $\phi(\lambda) = (\lambda - \lambda_0)^3$. There can be following possibilities:

(1) λ_0 has three linearly independent eigenvectors: $\mathrm{v}_1 = \begin{bmatrix} v_{11} \\ v_{21} \\ v_{31} \end{bmatrix}$, $\mathrm{v}_2 = \begin{bmatrix} v_{12} \\ v_{22} \\ v_{32} \end{bmatrix}$, and $\mathrm{v}_3 = \begin{bmatrix} v_{13} \\ v_{23} \\ v_{33} \end{bmatrix}$, then the general solution is

$$X_c(t) = (c_1\mathrm{v}_1 + c_2\mathrm{v}_2 + c_3\mathrm{v}_3)e^{\lambda_0 t},$$

and the fundamental matrix is

$$\Psi(t) = e^{\lambda_0 t}\begin{bmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{bmatrix}$$

λ_0 has two linearly independent eigenvectors. Suppose $\mathrm{v}_1 = \begin{bmatrix} v_{11} \\ v_{21} \\ v_{31} \end{bmatrix}$, and $\mathrm{v}_2 = \begin{bmatrix} v_{12} \\ v_{22} \\ v_{32} \end{bmatrix}$ are linearly independent eigenvectors. Let $\mathrm{v}_3 = \begin{bmatrix} v_{13} \\ v_{23} \\ v_{33} \end{bmatrix}$, then only one of the two equations $(\Theta - \lambda_0 I)\mathrm{v}_3 = \mathrm{v}_1$, or $(\Theta - \lambda_0 I)\mathrm{v}_3 = \mathrm{v}_2$, can have a solution that is linearly independent with v_1, v_2. Suppose $(\Theta - \lambda_0 I)\mathrm{v}_3 = \mathrm{v}_2$ generates such a solution, then the general solution is

$$X_c(t) = [c_1\mathrm{v}_1 + c_2\mathrm{v}_2 + c_3(t\mathrm{v}_2 + \mathrm{v}_3)]e^{\lambda_0 t},$$

and the fundamental matrix is

$$\Psi(t) = e^{\lambda_0 t}\begin{bmatrix} v_{11} & v_{12} & tv_{12} + v_{13} \\ v_{21} & v_{22} & tv_{22} + v_{23} \\ v_{31} & v_{32} & tv_{32} + v_{33} \end{bmatrix}$$

(3) λ_0 has only one eigenvector. Let $\mathrm{v}_1 = \begin{bmatrix} v_{11} \\ v_{21} \\ v_{31} \end{bmatrix}$ be the linearly independent eigenvector; and $\mathrm{v}_2 = \begin{bmatrix} v_{12} \\ v_{22} \\ v_{32} \end{bmatrix}$, and $\mathrm{v}_3 = \begin{bmatrix} v_{13} \\ v_{23} \\ v_{33} \end{bmatrix}$ be two vectors that satisfies, $(\Theta - \lambda_0 I)\mathrm{v}_2 = \mathrm{v}_1$, and $(\Theta - \lambda_0 I)\mathrm{v}_3 = \mathrm{v}_2$. Then the general solution is

$$X_c(t) = \left[c_1\mathrm{v}_1 + c_2(t\mathrm{v}_1 + \mathrm{v}_2) + c_3(t^2\mathrm{v}_1 + t\mathrm{v}_2 + \mathrm{v}_3)\right]e^{\lambda_0 t},$$

and the fundamental matrix is

$$\Psi(t) = e^{\lambda_0 t}\begin{bmatrix} v_{11} & tv_{11} + v_{12} & t^2v_{11} + tv_{12} + v_{13} \\ v_{21} & tv_{21} + v_{22} & t^2v_{21} + tv_{22} + v_{23} \\ v_{31} & tv_{31} + v_{32} & t^2v_{31} + tv_{32} + v_{33} \end{bmatrix}$$

The solution consists of expressions for change in prices with respect to their initial values before shocks for all three markets as a function of time, i.e., full dynamic adjustment paths for all prices when markets are out of equilibrium as well as their values when markets are in equilibrium. Initial and final equilibrium values, i.e., before and after shocks are obtained by substituting $t = 0$, and $t = \infty$, respectively. Various policy responses can be predicted and optimal policies can be derived by modeling peculiar types of shocks to markets.

4 CONCLUSION

A theoretical dynamical macroeconomic model based on interacting markets has been developed taking into consideration three markets, i.e., goods, capital/money, and labor markets. It provides results which are robust to heterogeneous consumers and producers exhibiting bounded rationality. Dynamic optimization problems of agents in all three markets have been solved to find expressions regarding their individual decisions, which have been solved simultaneously to get a nonhomogeneous linear system of differential equations, for which conditions for a unique solution has been specified. Also, conditions regarding stability and existence of an equilibrium have been stipulated. After estimating and substituting empirical parameter values in the system of differential equations and solving them, the response of three markets can be predicted. The model captures not only both initial and final sets of equilibria before and after shocks to all markets, rather it predicts the full adjustment path of all markets from initial to final equilibriums after various kinds of shocks happen to one or more markets. Furthermore, optimal policies, such as monetary policy, taxation, inflation control, employment, trade, remittances, etc., affecting one or more of the three markets subject to relevant constraints can be derived based on system of differential equations. The methodology employed for three markets can be extended to n number of markets in an economy.

5 FUTURE RESEARCH PROSPECTS

Optimal policies, such as monetary policy, taxation, inflation control, employment, trade, remittances, etc., affecting one or more of the three markets subject to relevant constraints can be derived.

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Conflicts of Interest
The authors declared no conflict of interest.

Author Contribution
Muhammad Ashfaq Ahmed conceived the main idea of the paper, planned on methodology, did literature review, sketched outlines, and details of the model. Nasreen Nawaz worked on mathematical derivations and solution of the model. Both authors jointly prepared the working draft of the article, proofread, and agreed on the final draft for submission to the journal.

Abbreviation List
ACE, Agent-based computational economics
DSGE, Dynamic stochastic general equilibrium
NK, New keynesian
NNS, New neoclassical synthesis

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